A proof of the prime number theorem

Hidetomo Tohmori　2022/10/03

Abstract

Previously, the prime number theorem was proved by directly calculating the probability that the natural number p is prime.

In this paper, the existence ratio of prime p among all natural numbers, the existence probability of prime p, is obtained from the existence ratio of prime p among natural numbers. This existence probability of prime p is the same as the probability that the natural number p is prime.

Introduction

Prime number p exists infinitely in all natural numbers in the form of its power p^n. However, the existence ratio of prime p among all natural numbers can be obtained from the existence ratio of prime p in a natural number. This existence ratio of prime p in all natural numbers is the existence probability of prime p in all natural numbers which is the same as the probability that the natural number p is prime.

Proof

There are always integer exponents n and m that hold the following inequality for every prime number p.

e is Napier's constant

$$e^m < p^n < e^{m+1}$$

Converting to logarithm gives the following inequality.

$$m < n \ln p < (m+1) \qquad \ln e = 1$$

$$1 < \frac{n \ln p}{m} < (1 + \frac{1}{m})$$

When the natural number m goes to infinity, the following equality holds.

$$\lim_{m \to \infty} \frac{n \ln p}{m} = 1$$

$$\lim_{m\to\infty}\frac{n}{m}=\frac{1}{\ln p}$$

Powers of prime number p and powers of e exist in the natural numbers e^{m+1} as below.

$$p^n \cong e^m$$

$$p^1 \quad p^2 \quad p^3 \quad p^4 \ldots\ldots\ldots\ldots \quad p^{n-3} \quad p^{n-2} \quad p^{n-1} \quad p^n$$

$$e^1 \quad e^2 \quad e^3 \quad e^4 \quad e^5 \quad e^6 \quad \ldots\ldots\ldots\ldots e^{m-5} \quad e^{m-4} \quad e^{m-3} \quad e^{m-2} \quad e^{m-1} \quad e^m$$

The exponent ratio $\frac{n}{m}$ is the ratio of the number n of powers of prime p to the number m of powers of e in natural number e^m.

Then the exponent ratio $\frac{n}{m}$ is the existence ratio of prime p in natural numbers e^m.

Then $\lim_{m\to\infty}\frac{n}{m}$ is the existence ratio of prime p in all natural numbers.

However, the above "the existence ratio of prime p in all natural numbers" is "the existence probability of prime p in all natural numbers".

Therefore, the existence probability of prime p is $\frac{1}{\ln p}$.

And the existence probability $\frac{1}{\ln p}$ is the same as the probability that the natural number p is prime.

Derivation of the prime counting function $\pi(x)$

Then, the prime counting function $\pi(x)$ is obtained by integrating above probability $\frac{1}{\ln p}$ that the natural number p is prime.

$$\pi(x)=\int_2^x \frac{1}{\ln p}dp \cong \frac{x}{\ln x}$$

As mentioned above, the prime number theorem has been proved.

References

1. Erdős, Paul (1949-07-01), "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem,", Proceedings of the National Academy of Sciences (U.S.A.: National Academy of Sciences) 35 (7): 374-384, doi:10.1073/pnas.35.7.374

supplement

$$e^m < p_1^{n_1}\ p_2^{n_2}\ p_3^{n_3}\ p_4^{n_4}\ p_5^{n_5} \dots\dots p_i^{n_i} \dots\dots p_{l-4}^{n_{l-4}}\ p_{l-3}^{n_{l-3}}\ p_{l-2}^{n_{l-2}}\ p_{l-1}^{n_{l-1}}\ p_l^{n_l} < e^{m+k}$$

For every power $p_i^{n_i}$ between natural numbers e^m and e^{m+k}, there must exist natural numbers m and k that satisfy the following inequality.

$$1 < \frac{n_i \ln p_i}{m} < (1 + \frac{k}{m})$$

Converting to logarithm gives the following inequality.

$$\frac{1}{\ln p_i} < \frac{n_i}{m} < (1 + \frac{k}{m})\frac{1}{\ln p_i}$$

$$\sum_{i=1}^{l} \frac{1}{\ln p_i} < \ \sum_{i=1}^{l} \frac{n_i}{m} \ < (1 + \frac{k}{m}) \sum_{i=1}^{l} \frac{1}{\ln p_i}$$

Since $\frac{n_i}{m}$ is the probability of existence of prime numbers in natural number e^{m+k}, the following equation holds.

$$\sum_{i=1}^{l} \frac{n_i}{m} = \sum_{i=1}^{l} \frac{1}{\ln p_i} \cong 1$$

$$\sum_{i=1}^{l} n_i \cong m$$