A proof of the prime number theorem

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Abstract

The prime p exists infinitely among all natural numbers in the form of powers p^n.

However, the existence ratio of the powers p^n among all natural numbers is $\frac{1}{\ln p}$.

Therefore, the prime p exists among all natural numbers with the existence ratio $\frac{1}{\ln p}$.

On the other hand, the natural number p exists in all natural numbers as the prime number p or the composite number p.

Then, the natural number p exists as the prime number p in all natural numbers with the existence ratio $\frac{1}{\ln p}$.

In other words, the natural number p is the prime number p with the existence ratio (probability) $\frac{1}{\ln p}$.

Therefore, the prime counting function $\pi(x)$ is obtained by integration with the probability $\frac{1}{\ln p}$.

Introduction

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Therefore, the prime counting function $\pi(x)$ is obtained by integration with the probability $\frac{1}{\ln p}$.

Derivation of the existence ratio $\frac{1}{\ln p}$

There are always natural number m and n that satisfies the following inequality for power p^n of prime p.

$$e^m < p^n < e^{m+1} \qquad e \text{ is Napier's constant}$$

Converting to logarithm gives the following inequality.

$$m < n \ln p < (m+1) \qquad \ln e = 1$$

$$1 < \frac{n \ln p}{m} < (1 + \frac{1}{m})$$

When the natural number m goes to infinity, the following equality holds.

$$\lim_{m \to \infty} \frac{n \ln p}{m} = 1$$

$$\lim_{m \to \infty} \frac{n}{m} = \frac{1}{\ln p}$$

The powers p^n of prime p and the powers e^m of e exist as below.

$$e^m < p^n < e^{m+1}$$

$$p^1 \;\; p^2 \;\; p^3 \ldots \;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\; p^n \;\; p^{n-1} \;\; p^n$$

$$e^1 \;\; e^2 \;\; e^3 \;\; e^4 \;\; e^5 \;\; e^6 \;\ldots\ldots\;\ldots\ldots\;\ldots\ldots\;\ldots\ldots\; e^{m-5} \;\; e^{m-4} \;\; e^{m-3} \;\; e^{m-2} \;\; e^{m-1} \;\; e^m$$

Therefore, the existence ratio $\frac{n}{m}$ is the existence ratio of prime p among natural numbers e^m.

Then $\lim_{m \to \infty} \frac{n}{m}$ is the existence ratio $\frac{1}{\ln p}$ of the prime p among all natural numbers.

On the other hand, the natural number p exists among all natural numbers as the prime p or the composite number p.

Then the natural number p exists as the prime p among all natural numbers with the existence ratio $\frac{1}{\ln p}$.

Simply put, the natural number p is the prime p with the probability $\frac{1}{\ln p}$.

Derivation of the prime counting function $\pi(x)$

The prime counting function $\pi(x)$ is obtained by integration with the probability $\frac{1}{\ln p}$ as below.

$$\frac{d}{dp}\left(\frac{p}{\ln p}\right) = \frac{1}{\ln p} - \frac{p}{(\ln p)^2}\frac{1}{p}$$

$$= \frac{1}{\ln p} - \frac{1}{(\ln p)^2}$$

$$\frac{1}{\ln p} = \frac{d}{dp}\left(\frac{p}{\ln p}\right) + \frac{1}{(\ln p)^2}$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}\,dp = \int_2^x \frac{d}{dp}\left(\frac{p}{\ln p}\right)dp + \int_2^x \frac{1}{(\ln p)^2}\,dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}\,dp = \frac{x}{\ln x} + \int_2^x \frac{1}{(\ln p)^2}\,dp$$

$$\int_2^x \frac{1}{(\ln p)^2}\,dp = \frac{x}{(\ln x)^2} + 2\int_2^x \frac{1}{(\ln p)^3}\,dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}\,dp = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + 2\int_2^x \frac{1}{(\ln p)^3}\,dp$$

$$2\int_2^x \frac{1}{(\ln p)^3}\,dp = \frac{2x}{(\ln x)^3} + 6\int_2^x \frac{1}{(\ln p)^4}\,dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}\,dp = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + 6\int_2^x \frac{1}{(\ln p)^4}\,dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}\,dp = \frac{x}{\ln x} + \sum_{n=1}^{m} \frac{n!x}{(\ln x)^{n+1}}$$

$$\frac{d}{dx}\left(\frac{m!x}{(\ln x)^{m+1}}\right) = \frac{m!}{(\ln x)^{m+1}} - \frac{(m+1)!x}{(\ln x)^{m+2}}\frac{1}{x} = 0$$

$$1-\frac{m+1}{\ln x}=0 \qquad m=\ln x-1$$

As described above, the approximation of the above prime counting function $\pi(x)$ is possible with scientific calculator without computer.

Five examples are presented below.

$$x=10^2 \qquad m=\ln x-1=\ln 10^2-1=3.$$

$$\pi(x)-\frac{x}{\ln x}=3. \text{ (Wikipedia)}$$

$$\frac{x}{(\ln x)^2}=\frac{10^2}{(\ln 10^2)^2}=4.$$

$$\frac{x}{(\ln x)^2}+\frac{2x}{(\ln x)^3}=\frac{10^2}{(\ln 10^2)^2}+\frac{2\times 10^2}{(\ln 10^2)^3}=6.$$

$$x=10^3 \qquad m=\ln x-1=\ln 10^3-1=5.$$

$$\pi(x)-\frac{x}{\ln x}=23. \text{ (Wikipedia)}$$

$$\frac{x}{(\ln x)^2}=\frac{10^3}{(\ln 10^3)^2}=20.$$

$$\frac{x}{(\ln x)^2}+\frac{2x}{(\ln x)^3}=\frac{10^3}{(\ln 10^3)^2}+\frac{2\times 10^3}{(\ln 10^3)^3}=27.$$

$$x=10^{10} \qquad m=\ln x-1=\ln 10^{10}-1=22.$$

$$\text{(Wikipedia)} \quad \pi(x)-\frac{x}{\ln x}=20{,}758{,}029.$$

$$\frac{x}{(\ln x)^2}=\frac{10^{10}}{(\ln 10^{10})^2}=18{,}861{,}169.$$

$$\frac{x}{(\ln x)^2}+\frac{2x}{(\ln x)^3}=\frac{10^{10}}{(\ln 10^{10})^2}+\frac{2\times 10^{10}}{(\ln 10^{10})^3}=20{,}499{,}430.$$

$$\frac{x}{(\ln x)^2}+\frac{2x}{(\ln x)^3}+\frac{6x}{(\ln x)^4}=\frac{10^{10}}{(\ln 10^{10})^2}+\frac{2\times 10^{10}}{(\ln 10^{10})^3}+\frac{6\times 10^{10}}{(\ln 10^{10})^4}=20{,}712{,}876.$$

$$\frac{x}{(\ln x)^2}+\frac{2x}{(\ln x)^3}+\frac{6x}{(\ln x)^4}=\frac{10^{10}}{(\ln 10^{10})^2}+\cdots+\frac{24\times 10^{10}}{(\ln 10^{10})^5}=20{,}749{,}955.$$

$$\frac{x}{(\ln x)^2}+\frac{2x}{(\ln x)^3}+\frac{6x}{(\ln x)^4}=\frac{10^{10}}{(\ln 10^{10})^2}+\cdots+\frac{120\times 10^{10}}{(\ln 10^{10})^6}=20{,}758{,}006.$$

$$\frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + \frac{6x}{(\ln x)^4} = \frac{10^{10}}{(\ln 10^{10})^2} + \cdots + \frac{720 \times 10^{10}}{(\ln 10^{10})^7} = 20{,}760{,}104.$$

$$\frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + \frac{6x}{(\ln x)^4} = \frac{10^{10}}{(\ln 10^{10})^2} + \cdots + \frac{5040 \times 10^{10}}{(\ln 10^{10})^8} = 20{,}760{,}741.$$

$x = 10^{20} \qquad m = \ln x - 1 = \ln 10^{20} - 1 = 45.$

(Wikipedia) $\pi(x) - \frac{x}{\ln x} = 49{,}347{,}193{,}044{,}659{,}701.$

$$\frac{x}{(\ln x)^2} = \frac{10^{20}}{(\ln 10^{20})^2} = 47{,}152{,}924{,}252{,}903{,}482.$$

$$\frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} = \frac{10^{20}}{(\ln 10^{20})^2} + \frac{2 \times 10^{20}}{(\ln 10^{20})^3} = 49{,}200{,}749{,}733{,}767{,}281.$$

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$x = 10^{28} \qquad m = \ln x - 1 = \ln 10^{28} - 1 = 63.$

(Wikipedia) $\pi(x) - \frac{x}{\ln x} = 2{,}484{,}097{,}167{,}669{,}186{,}251{,}622{,}127.$

$$\frac{x}{(\ln x)^2} = \frac{10^{28}}{(\ln 10^{28})^2} = 2{,}405{,}761{,}441{,}474{,}667{,}464{,}540{,}316.$$

$$\frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} = \frac{10^{28}}{(\ln 10^{28})^2} + \frac{2 \times 10^{28}}{(\ln 10^{28})^3} = 2{,}480{,}390{,}649{,}960{,}957{,}535{,}973{,}511.$$

$$\frac{x}{(\ln x)^2} \ldots + \frac{6x}{(\ln x)^4} = \frac{10^{28}}{(\ln 10^{28})^2} \ldots + \frac{6 \times 10^{28}}{(\ln 10^{28})^4} = 2{,}483{,}863{,}262{,}828{,}929{,}297{,}882{,}432.$$

$$\frac{x}{(\ln x)^2} \dots + \frac{6x}{(\ln x)^4} = \frac{10^{28}}{(\ln 10^{28})^2} \dots + \frac{24\times 10^{28}}{(\ln 10^{28})^5} = 2{,}483{,}917{,}124{,}850{,}584{,}525{,}088{,}882.$$

$$\frac{x}{(\ln x)^2} \dots + \frac{6x}{(\ln x)^4} = \frac{10^{28}}{(\ln 10^{28})^2} \dots + \frac{120\times 10^{28}}{(\ln 10^{28})^6} = 2{,}483{,}933{,}833{,}406{,}862{,}395{,}538{,}927.$$

$$\frac{x}{(\ln x)^2} \dots + \frac{6x}{(\ln x)^4} = \frac{10^{28}}{(\ln 10^{28})^2} \dots + \frac{720\times 10^{28}}{(\ln 10^{28})^7} = 2{,}483{,}935{,}388{,}356{,}960{,}691{,}768{,}830.$$

As shown in the above five examples, when $x < 10^{10}$, $\pi(x)$ is over-prime-counted.

However, for $x > 10^{10}$, computing up to $m = \ln x - 1$, $\pi(x)$ is expected to be the actual prime factor.

As mentioned above, it is proved that the existence ratio $\frac{1}{\ln p}$ is the probability that the natural number p is prime number p.

References

1. Erdős, Paul (1949-07-01), "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem,", Proceedings of the National Academy of Sciences (U.S.A.: National Academy of Sciences) 35 (7): 374-384, doi:10.1073/pnas.35.7.374