ABC conjecture

Hidetomo Tohmori 2023.10.01

ABC conjecture states that there are natural numbers A, B and C that satisfy equation (1.1) and inequality (1.2) below, but there are no natural numbers A, B and C that satisfy inequality (1.3).

p is a prime number (≥ 5) r is a natural number

Natural numbers A, B and C are relatively prime

$$A + B = C \qquad (1.1)$$

$$rad(ABC) < C \qquad (1.2)$$

$$rad(ABC)^2 < C \qquad (1.3)$$

$$A = rad(A) = p = C - B \qquad rad(BC) = r$$

For inequality (1.2) to hold, the following inequality must hold.

$$rad(ABC) = pr < C \qquad p < \frac{1}{r}C$$

For inequality (1.3) to hold, the following inequality must hold.

$$rad(ABC)^2 = p^2r^2 < C \qquad p < \frac{1}{r}C^{\frac{1}{2}}$$

The maximum value of prime number p that satisfies equation (1.1) and inequalities (1.2) and (1.3) is determined by the minimum value of $r = rad(BC)$.

However, the minimum value of r is 6 when B and C are powers of 2 or 3.

Then, the equation (1.1) can be rewritten as follows.

$$p + 3^j = 2^n \qquad p + 2^n = 3^j \qquad (1.4)$$

In order for the inequalities (1.2) and (1.3) to hold, the following inequalities (1.5) and (1.6) must hold.

$$rad(BC) = 2 \times 3 = r = 6$$

$$rad(ABC) = 6p < C \qquad p < \frac{1}{6}C \qquad (1.5)$$

$$rad(ABC)^2 = 36p^2 < C \qquad p < \frac{1}{6}C^{\frac{1}{2}} \qquad (1.6)$$

Then, ABC conjecture can be rephrased as follows.

There are prime number p and natural numbers j and n that satisfy equation (1.4) and inequality (1.5) above, but there are no prime number p and natural numbers j and n that satisfy inequality (1.6).

The above ABC conjecture is proven below.

There always exist real numbers m, δ and α that satisfy the following inequality.

$$e^m < B < e^{m+\delta} \qquad e^{m+\alpha} < C < e^{m+\alpha+\delta}$$

Then the following inequality holds.

$$e^{m+\alpha} - e^{m+\delta} < C - B = p < e^{m+\alpha+\delta} - e^m$$

$$e^m(e^\alpha - e^\delta) < p < e^m(e^{\alpha+\delta} - 1)$$

$$0 < p = C - B \qquad e^\alpha - e^\delta = \beta > 0 \qquad \beta \text{ is a real number.}$$

$$\beta e^m < p < e^m(e^{\alpha+\delta} - 1)$$

When δ is extremely small, the following inequality holds.

$$\beta e^m \approx \beta B < p < e^m(e^{\alpha+\delta} - 1)$$

Then the following inequality holds.

$$\beta B = \beta(C - p) < p$$

$$\beta C < p + \beta p = (1 + \beta)p$$

$$\frac{\beta}{1+\beta} C < p$$

However, when ABC conjecture holds, the following inequality (1.6) does not hold.

$$rad(ABC)^2 = 36p^2 < C \qquad p < \frac{1}{6} C^{\frac{1}{2}} \qquad (1.6)$$

Then the following inequality does not hold.

$$\frac{\beta}{1+\beta} C < \frac{1}{6} C^{\frac{1}{2}}$$

$$(\frac{\beta}{1+\beta})^2 C^2 < \frac{1}{36} C$$

$$C < \frac{1}{36} (\frac{1+\beta}{\beta})^2$$

Since there is no limit to the size of natural number C, the above inequality does not hold.

Therefore, ABC conjecture holds.

In the following five examples, the inequality (1.2) holds, but the inequality (1.3) does not.

From these five examples, ABC conjecture is presumed to be reasonable.

Example 1

$A = 5 \qquad B = 3^3 = 27 \qquad C = 2^5 = 32$

$rad(A) = 5 \qquad rad(B) = 3 \qquad rad(C) = 2$

$rad(ABC) = 5 \times 2 \times 3 = 30 < 32 = 2^5 = C$

Example 2

$A = 13 \qquad B = 3^5 = 243 \qquad C = 2^8 = 256$

$rad(A) = 13 \qquad rad(B) = 3 \qquad rad(C) = 2$

$rad(ABC) = 13 \times 2 \times 3 = 78 < 256 = 2^8 = C$

Example 3

$A = 139 \qquad B = 2^{11} = 2048 \qquad C = 3^7 = 2187$

$rad(A) = 139 \qquad rad(B) = 2 \qquad rad(C) = 3$

$rad(ABC) = 139 \times 2 \times 3 = 834 < 2187 = 3^7 = C$

Example 4

$A = 6487 \qquad B = 3^{10} = 59049 \qquad C = 2^{16} = 65536$

$rad(A) = 6487 \qquad rad(B) = 3 \qquad rad(C) = 2$

$rad(ABC) = 6487 \times 3 \times 2 = 38922 < 65536 = 2^{16} = C$

Example 5

$A = 7153 \qquad B = 2^{19} = 524288 \qquad C = 3^{12} = 531441$

$rad(A) = 7153 \qquad rad(B) = 2 \qquad rad(C) = 3$

$rad(ABC) = 7153 \times 2 \times 3 = 42918 < 531441 = 3^{12} = C$