# Goldbach’s conjecture

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**Abstract**

Two sequences containing all prime numbers is generated.

Any even number is generated by summing any two numbers in two sequences.

The probability that both of above two numbers are prime is given by the prime number theorem.

And the sum of the probabilities diverges to infinity.

That is, it proves that every even number can be represented by the sum of two primes.

In addition, the prime number theorem is proved probabilistically.

## 1. Generation of two sequences containing every prime number

The formula for the sum of squares of natural numbers is as follows.

$\sum_{i=1}^{k} i^2 = 1 + 2^2 + 3^2 + \cdots + (k-2)^2 + (k-1)^2 + k^2 = k(K+1)(2k+1)/6$

Since the sum of squares of natural numbers is an integer, $k(K+1)(2k+1)$ is divisible by the number $6$.

When $(2k+1)$ is a prime number, $(2k+1)$ is indivisible by number $6$. At that time, $k$ or $k+1$ is a multiple of the number $3$ because either is an even number. $m$ is a natural number.

$$k = 3m \text{ or } k+1 = 3m \qquad p_m = 2k+1 = 6m \pm 1$$

## 2. Generation of any even number

The following $p_m$ and $p_n$ are composite numbers or prime numbers. $n$ is a natural number.

$$p_m = 6m \pm 1 \qquad p_n = 6n \pm 1 \qquad m + n = l$$

$$\begin{aligned} p_m + p_n = 6m \pm 1 + 6n \pm 1 &= 6(m+n) - 2 = 6l - 2 \\ &= 6(m+n) \qquad = 6l \\ &= 6(m+n) + 2 = 6l + 2 \end{aligned}$$

As mentioned above, any even number greater than or equal to $10$ $(l \geq 2)$ can be generated by the sum of $p_m$ and $p_n$.

## 3. Probabilistic proof

Using the prime number theorem "The probability that natural number $x$ is prime number is $(\frac{1}{\ln x})$", the probability $q$ that both $p_m$ and $p_n$ are prime numbers is as follows.

$$q = (\frac{1}{\ln 6m})(\frac{1}{\ln 6(l-m)})$$

Accumulating gives the following inequality.

$$\sum q = \sum_{m=1}^{l}(\frac{1}{\ln 6m})(\frac{1}{\ln 6(l-m)}) > \frac{l}{(\ln l)^2}$$

$\frac{l}{(\ln l)^2}$ on the right side diverges to infinity as $l$ increases.

Then the probability $\sum q$ naturally diverges to infinity.

As mentioned above, the Goldbach conjecture was proved based on the prime number theorem.

Below is a probabilistic proof of the prime number theorem.

## 4 A proof of the prime number theorem

### At the beginning

When a natural number $p$ is a prime number $p$, the prime number $p$ exists infinitely in all natural numbers in the form of power $p^n$.

However, the existence probability $\frac{1}{\ln p}$ of power $p^n$ in all natural numbers can be obtained from the existence probability of power $p^n$ in natural number $e^m$.

And the existence probability $\frac{1}{\ln p}$ of the power $p^n$ is the existence probability $\frac{1}{\ln p}$ of the prime number $p$.

Then every natural number is prime $p$ with existence probability $\frac{1}{\ln p}$.

That is, natural number $p$ is prime number $p$ with the existence probability $\frac{1}{\ln p}$.

That is, the existence probability $\frac{1}{\ln p}$ is the probability that natural number $p$ is prime.

The prime counting function $\pi(x)$ is obtained by integration with probability $\frac{1}{\ln p}$.

**Derivation of $\frac{1}{\ln p}$**

There are always natural numbers $n$ and $m$ that hold the following inequality for every prime number $p$.

$e$ is Napier's constant

$$e^m < p^n < e^{m+1}$$

Converting to logarithm, the following inequality holds.

$$m < n \ln p < (m+1) \qquad \ln e = 1$$

$$1 < \frac{n \ln p}{m} < (1 + \frac{1}{m})$$

When the natural number $m$ goes to infinity, the following equality holds.

$$\lim_{m\to\infty} \frac{n \ln p}{m} = 1$$

$$\lim_{m\to\infty} \frac{n}{m} = \frac{1}{\ln p}$$

Power $p^n$ and power $e^m$ exist in natural numbers $e^m$ as follows.

$$e^m < p^n < e^{m+1}$$

$$p_l{}^1$$

$$p_{l-1}{}^1 \ \ p_{l-1}{}^2$$

$$p_{l-2}{}^1 \ \ p_{l-2}{}^2 \ \ p_{l-2}{}^3$$

$$\dots \ \dots \ \dots \ \dots \ \dots$$

$$p^1 \ \ p^2 \ \ p^3 \dots \ \dots \ \dots \ \dots \ \dots \ p^n \ \ p^{n-1} \ \ p^n$$

$$\dots \ \dots \ \dots \ \dots \ \dots$$

$$p_3{}^1 \ \ p_3{}^2 \ \ p_3{}^3 \ \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ \ p_3{}^{j-2} \ \ p_3{}^{j-1} \ \ p_3{}^j$$

$$p_2{}^1 \ \ p_2{}^2 \ \ p_2{}^3 \ \ p_2{}^4 \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ p_2{}^{k-3} \ \ p_2{}^{k-2} \ \ p_2{}^{k-1} \ \ p_2{}^k$$

$$p_1{}^1 \ \ p_1{}^2 \ \ p_1{}^3 \ \ p_1{}^4 \ \ p_1{}^5 \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ p_1{}^{i-4} \ \ p_1{}^{i-3} \ \ p_1{}^{i-2} \ \ p_1{}^{i-1} \ \ p_1{}^i$$

$$e^1 \ \ e^2 \ \ e^3 \ \ e^4 \ \ e^5 \ \ e^6 \ \ \dots\dots \ \ \dots\dots \ \ \dots\dots \ \ \dots\dots \ \ e^{m-5} \ \ e^{m-4} \ \ e^{m-3} \ \ e^{m-2} \ \ e^{m-1} \ \ e^m$$

Therefore, the exponential ratio $\frac{n}{m}$ is the existence probability of prime $p$ among the natural numbers $e^m$.

Then $\lim_{m\to\infty}\frac{n}{m}$ is the existence probability $\frac{1}{\ln p}$ of prime $p$ among all natural numbers.

And since the prime $p$ exists among all natural numbers with the existence probability $\frac{1}{\ln p}$, every natural number is prime $p$ with the existence probability $\frac{1}{\ln p}$.

That is, the existence probability $\frac{1}{\ln p}$ is the probability that natural number $p$ is prime.

**Derivation of the prime counting function $\pi(x)$**

The prime counting function $\pi(x)$ is obtained by integration with the probability $\frac{1}{\ln p}$ as follows.

$$\pi(x) = \int_2^x \left(\frac{1}{\ln p}\right) dp \cong x/\ln x$$

Thus, the prime number theorem has been proved.

**References**

1. Erdős, Paul (1949-07-01), "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem,", Proceedings of the National Academy of Sciences (U.S.A.: National Academy of Sciences) 35 (7): 374-384, doi:10.1073/pnas.35.7.374