A new method for calculating the bending angle of light in a gravitational field

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Abstract

Schwarzschild solution has been used to calculate light bending angle in a gravitational field by performing an elliptic integration. However, because Schwarzschild solution includes a non-linear distortion of the radial axis, it is very difficult to calculate the bending angle. Herein a new method to calculate this quantity was developed based on the assumptions that both of the speed and the direction of propagation of light change in a gravitational field. The validity of the new method is proved by showing that the bending angle produced by the non-linear distortion of the radial axis in Schwarzschild solution corresponds to that obtained from the assumptions of the new method. Consequently, the new method is identical to Schwarzschild solution, but simple, easy and exact.

1. Introduction

In this study, a new method for calculating the bending angle of light in a gravitational field is proposed. The new method is based on the assumptions that the radial axis is not distorted by the field but that light propagates changing its speed and direction in a gravitational field. In other words, the non-linear distortion of the radial axis in Schwarzschild solution corresponds to a variation of the bending angle of light, which is calculated using the new method.

2. Relation between Schwarzschild solution and the new method

Consider a spherically symmetric coordinate system with a point mass M at the center. Let R' be the radial coordinate in this system. Furthermore, the following quantities are defined.

C : Speed of light in a static system

C' : Speed of light in a gravitational field

θ : Rotation angle of light around the center

ω : Rotation angle of the direction of light propagation

d : Infinitesimal change i.e. differential

G : Gravitational constant

We also define the parameter $\alpha = 2GM/C^2$.

Schwarzschild solution is as the following.

(1) $-dS^2 = C^2(1- \alpha/R')dt'^2 - dR'^2/(1 - \alpha/R') - R'^2(d\theta^2 + \sin^2\theta d\Phi^2)$

Where $\alpha = 2GM/C^2$, $C'^2 = C^2(1- \alpha/R')$ and $d\Phi = 0$

With the parameters defined above, equation (1) can be rewritten as

(2) $-dS^2 + (\alpha/R')dR'^2/(1-\alpha/R') = C'^2dt'^2 - dR'^2 - R'^2d\theta^2$

The right-hand side of equation (2) vanishes, as shown in Fig. 1.

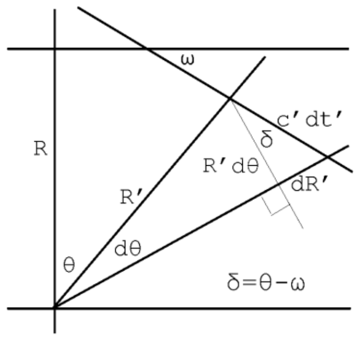

Fig. 1

We next define the quantity $(d\omega/d\theta)$ by the following equation:

(3) $(d\omega/d\theta) = (\alpha/R')/(1-\alpha/R')$

In section **4**, we also derive equation (3) from the assumptions of the new method.

Using equation (3), equation (2) can be rewritten as

(4) $-dS^2 + (d\omega/d\theta)dR'^2 = (1 -/R')dt'^2 - dR'^2 - R'^2d\theta^2 = 0$

Equation (4) demonstrates that the quantity $(d\omega/d\theta)$ corresponds to the non-linear distortion of the radial axis

R' in the original definition of dS^2. Note that the right-hand side of the equation (4) includes no distortion of the radial axis R'.

3. How the speed of light is changed in a gravitational field

It is assumed that an arbitrary location in gravitational fields belongs to an inertia system having time axis t' with speed of light C and radial axis R' moving with speed $U(R')$ relative to a static system. The static system is taken to have time axis t with speed of light C and radial axis R. Then, since the proper distance dS^2 is preserved under a coordinate transformation from the static system to the inertia system, we have the following formula.

Here, C' is the speed of light at an arbitrary position in the inertial system (i.e. a gravitational field), C is the speed of light in the static system (i.e. a non-gravitational field) and i is the imaginary unit ($i^2 = -1$).

$$dS^2 = (iCdt)^2 = dS'^2 = (iCdt')^2 + (U(R)dt')^2$$

$$d\theta = 0 , \quad d\Phi = 0$$

Note that the inertia system to which the arbitrary location belongs differs at different locations and that no gravitational field is present in the static system.

Since the distance through which light propagates is independent of the inertia system, we have the following equation.

$$C'dt' = Cdt.$$

This yields the following equation.

(6) $$C'^2 = C^2 - U^2 , \quad U = U(R)$$

When mass $m(\ll M)$ with speed v enters the gravitational field of a mass M from infinity, we have the following equations:

$$(iCdt')^2 + (Udt')^2 + (vdt')^2 = (dS')^2 = (iCdt')^2 + (V'dt')^2$$
$$(Udt')^2 + (vdt')^2 = (V'dt')^2$$
$$V'^2 = v^2 + U^2$$
$$mV'^2/2 = mv^2/2 + mU^2/2$$
$$d(mv^2/2)/dR = 0$$
$$d(mV'^2/2)/dR = d(mU^2/2)/dR = F = mg = mGM/R^2$$
$$dU^2/dR = 2GM/R^2$$
$$U^2 = 2GM/R$$

where $g = GM/R^2$ is the gravitational acceleration, v is the speed of mass m at infinity, V' is its speed in the gravitational field and F is the gravitational force acting on mass m.

Using these definitions, we can rewrite equation (6) as.

(7) $C'^2 = C^2 - 2GM/R$

Note that when $C'^2 = 0$, we have $C^2 - 2GM/Rs = 0$, where $Rs = 2GM/C^2$ is Schwarzschild radius

4. How does the propagation direction of light change in a gravitational field ?

In Fig.2, we assume a beam of light pass a mass M at a distance R traverse to radial direction. Light then continues past the direction defined by the angle θ at a distance R' from mass M with the direction of propagation given by the angle ω.

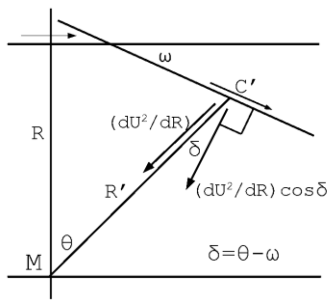

Fig.2

We assume that the angular rate of change $d\omega/dt'$ of the direction of propagation of light in the gravitational field varies according to

(8) $C'(d\omega/dt') = (dU^2/dR)\cos\delta$

$= C'(d\omega/d\theta)(d\theta/dt')$

where $\delta = \theta - \omega$.

As illustrated in Fig.3, we define

$$d\theta/dt' = C'\cos\delta/R'$$

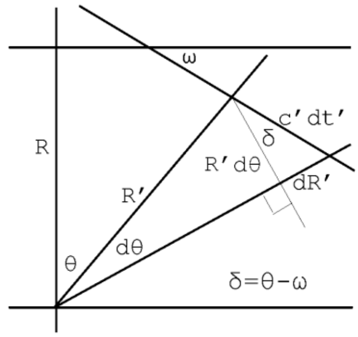

Fig.3

Using this equation, we can rewrite equation (8) as follows:

$$d\omega/dt' = (d\omega/d\theta)(C'\cos\delta/R') = 2GM\cos\delta/C'R'^2$$

$$d\omega/d\theta = 2GM/C'^2R'$$

Using equation (7), the equation (3) of section **2** can be rewritten as

(9) $d\omega/d\theta = (2GM/C^2R')/(1-2GM/C^2R')$

$= (Rs/R')/(1-Rs/R')$

where $Rs = 2GM/C^2$.

5. Specific examples of the bending angle of light in a gravitational field

5.1 Let us consider the case defined by $Rs \ll R = R'\cos\theta$, $\alpha/R' \ll 1$ and $\omega \ll \theta$. As shown in Fig.4, the following equations are formed.

$$d\omega/d\theta = (Rs/R)\cos\theta$$

$$\omega_p = \int d\omega = \int_{-\pi/2}^{\pi/2} \int (Rs/R)\cos\theta\, d\theta$$

$$= (Rs/R)(\sin(\pi/2) - \sin(-\pi/2))$$

$$= 2Rs/R' = 4GM/C^2R$$

The bending angle ω_p of light passing the periphery of the sun may be calculated using this equation.

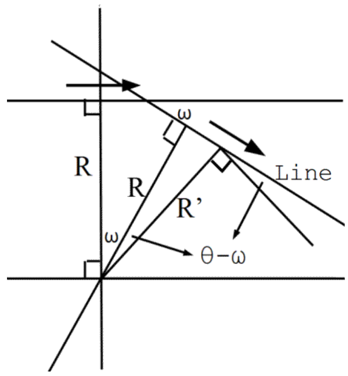

Fig.4

5.2 Next, consider the case defined by $R = \gamma Rs$, $2 < \gamma$, $\delta = \theta - \omega > 0$ and $d\delta/d\theta > 0$. This corresponds the following equations(Fig.5).

$$dR' = C'dt'\sin\delta$$

$d\theta/dt' = C'\cos\delta/R'$
$dR'/dt' = (dR'/d\theta)(d\theta/dt') = C'\sin\delta$
$(dR'/d\theta)(C'\cos\delta/R') = C'\sin\delta$
$d(\log R')/d\theta = \sin\delta/\cos\delta = -d(\log(\cos\delta))/d\delta$
$\log R' = -\log(\cos\delta) + \log A$,
A: Integration constant
$R'\cos\delta = R'\cos(\theta - \omega) = A$, $\theta = \omega$
$R' = R = A$
(10) $R'\cos\delta = R$

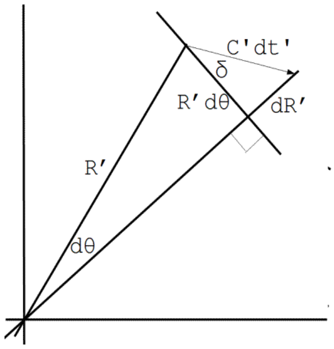

Fig.5

With these definitions, we can rewrite equation (9) as
(11) $d\omega/d\theta = Rs(\cos\delta/R)/(1 - Rs(\cos\delta/R))$,
where again $Rs = 2GM/C^2$. Equation (11) can also be written in the form
(12) $d\delta/d\theta = (\gamma - 2\cos\delta)/(\gamma - \cos\delta) > 0$,
which describes a parabolic path.
As shown by the thick line in Fig.6, angle δ increases as θ increases.
The fold point of the thick line moves upward or downward on the line of $\theta = \pi/2$ as γ increases or decreases.

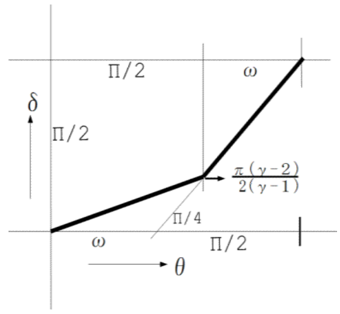

Fig.6

5.3 For the case defined by $1 < \gamma < 2$, $R = \gamma Rs$, $-\delta = \delta' = \omega - \theta > 0$, as shown in Fig. 7, we can write the following formulas:
$(dR'/d\theta)(C'\cos\delta'/R') = -C'\sin\delta$,
$d(\log R')/d\theta = -\sin\delta'/\cos\delta' = d(\log(\cos\delta'))/d\delta'$,
$\log R' = \log(\cos\delta') + \log A$,
where $\log A$ is a constant of integration,
$R' = A\cos\delta'$, $\theta = \omega$, $R' = R = A$ and
$R' = R\cos(\omega - \theta) = \gamma Rs(\cos\delta')$.

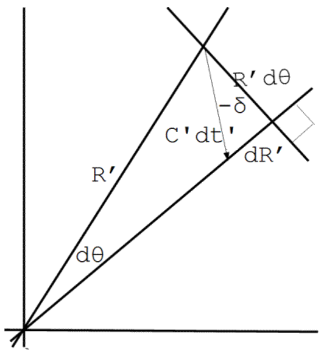

Fig.7

With these results, formula (6) can be rewritten as
(8) $d\omega/d\theta = (2GM/\cos(\omega - \theta)C^2R)/(1 - (2GM/\cos(\omega - \theta)C^2R))$ and
(9) $d\delta'/d\theta = (\gamma\cos\delta' - 2)/(\gamma\cos\delta' - 1) > 0$

which describes a spiral path.

5.4 In the case of $\gamma = 1$ and $R = Rs = 2GM/C2$, the following formulas are formed.

$$C'^2 = C^2 - 2GM/Rs = C'^2 - C^2 = 0$$

Therefore, the light could not exist in the case of $\gamma < 1$, because C'^2 cannot be minus.

6. Conclusions

In the new method described, it is assumed that light does not propagate in straight lines in a gravitational field. Instead, we assume that it propagates with changing speed C' defined by $C'^2 = C^2 - 2GM/R$, and that it bends in the direction corresponding to the acceleration defined by dU^2/dR $(U^2 = 2GM/R)$. Because the bending angle derived from these assumptions corresponds to the non-linear distortion of the radial axis in the Schwarzschild solution, the new method is the same as the Schwarzschild solution. However the new method is simple, easy to calculate and yields exactly the same bending angle as that given by the Schwarzschild solution.

References

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