

Peg Solitaire Diamond

Origin

Hana: Peg Solitaire Diamond game board is sold now by the name "FORTY-ONE" (registered trademark).

Taro: The manufacturer of the board must have named "FORTY-ONE" for Peg Solitaire Diamond.

Hana: "The game comes from the family crest of Shingen Takeda" is mentioned in the pamphlet of "FORTY-ONE".

Taro: In Europe, Peg Solitaire is known well from old days regardless of said family crest.

Taro: It is called Peg Solitaire in distinction from Solitaire of cards.

Hana: What word is the term "Solitaire"?

Taro: "Solitaire" is spelt solitaire in English.

Taro: The term "solitaire" means "one fixed accessories", "one solitaire card" or "a decoration consisting of one jewels".

Taro: Generally, it might be the meaning about "Only One Play".

Hana: How long is Peg Solitaire from old days?

Taro: "FORTY-ONE" is one that this manufacturer learnt from his grandfather and the grandfather talled him "Telling others is useless".

Hana: Mystery. It is being in Europe from old days.

Taro: It is said that the oldest record by paper documents in Europe is the letter by Leibnitz put out in 1710.

Hana: Is Leibnitz that famous mathematician?

Taro: It is so. It is Leibnitz who created the differentiation and the integration method independently of Newton.

Hana: For 2010, it is just 300 this year from 1710.

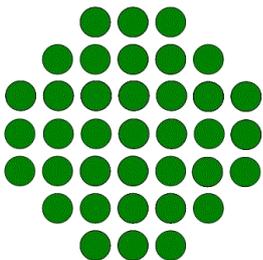
Taro: The signs that Leibnitz used to indicate the differentiation and integration are used now, but Newton's signs are not used.

Hana: What is written in the letter by Leibnitz?

Taro: The letter is written in Latin that Leibnitz enjoyed Peg Solitaire by the playing way that doesn't remove pegs (stones) but increases pegs oppositely.

Hana: Is the board of the Peg Solitaire the same as "FORTY-ONE".

Taro: Unfortunately it is the board with 33 holes



which is called "Peg Solitaire British".

Hana: Then, how different are "FORTY-ONE" and Peg Solitaire British ?

Taro: "FORTY-ONE" is one kind of Peg Solitaire games, too.

Taro: Besides FORTY-ONE, there is Peg Solitaire French with 37 holes, too.

Taro: We call Peg Solitaire British, French and Diamond each distinguishing in the future.

Taro: However, Peg Solitaire British is generally called "Peg Solitaire".

Taro: It is said that the French noble who was kept in the prison to kill time devised this Solitaire game.

Hana: I think that since this board game is very difficult, the intellectual prisoner like an aristocrat could kill time about.

Hana: Isn't there any record by documents as to when and by whom Peg Solitaire have been created.

Taro: I have watched a photograph of the slab that sculptures the appearance of the woman who is putting Peg Solitaire British board on her sideward table, the photograph was carried by the book.

Taro: And the production year 1697 could be read from the photograph.

Hana: In Europe, since Peg Solitaire is known well from old days, there must be a lot of documents that described about Peg Solitaire.

Taro: It is so. Anyway, because Peg Solitaire greatly became popular in the 18th century Europe and there was at least one Peg Solitaire board at each home.

Hana: Does not the document remain?

Taro: There is the book "Amusement mathematics" in which the person named Ryuka in France got the first edition in 1897.

Taro: A lot of famous mathematicians studied mathematics actively in France.

Hana: Mathematically thinking about Peg Solitaire seems to be interesting well.

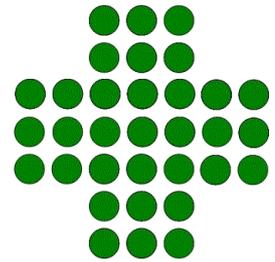
Taro: The solution method of Peg Solitaire British is mentioned considerably in detail in the "Amusement mathematics" of Ryuka.

Taro: However, in it, only one answer has been described about Peg Solitaire French and Peg Solitaire Diamond.

Taro: The peg array and the playing way of Peg Solitaire Diamond described in it are quite the same as those of FORTY-ONE.

Hana: Is it the book in which Peg Solitaire Diamond was first described ?

Taro: It might be so. Peg Solitaire French seems to have been enjoyed



in France. But It is not uncertain whether the board special of Peg Solitaire French was actually made.

Hana: Did Peg Solitaire French become popular, too?

Taro: Peg Solitaire French might be not familiar because it is very difficult as well as Peg Solitaire Diamond.

Taro: However, A lot of boards of Peg Solitaire British which was remodeled to be able to play Peg Solitaire French stay.

Taro: I think that the board which can play both Peg Solitaire British and 37 is sold in Japan.

Hana: I think that the manufacturing distributor of FORTY-ONE manufactured and sold the board of Peg Solitaire Diamond first.

Taro: The possibility is high.

Hana: In Japan, 1710 when Leibnitz's letter was mailed is Houei after Genroku Age in Edo period.

Hana: Isn't it possible that Peg Solitaire British or French board was transmitted and spread to Japan?

Taro: There is a possibility that the board was brought into Japan because Netherland was allowed to trade in Dejima in Nagasaki.

Taro: Actually, the play transmitted from Europe in Edo period is some known.

Hana: What play is it?

Taro: Mr. Shigeo Takagi talled me that the play which is a kind of "Circle of Wisdom" disappeared without so leaving signs when the fashion of the play ended, though the play greatly became popular on nationwide scale in Japan of Edo period.

Hana: It is possible that the crews of Dutch ship brought Peg Solitaire Diamond in.

Taro: Was it just good of the lengthy voyage to kill time?

Taro: It is unreasonable to assume that Peg Solitaire Diamond was just brought into Japan from Europe because Europe's was Peg Solitaire British or French.

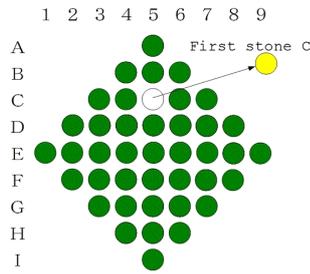
Hana: How about the idea that the grandfather of manufacturing distributor of FORTY-ONE created Peg Solitaire Diamond from Peg Solitaire British?

Taro: The possibility is also incontrovertible.

Taro: However, all answers of this Peg Solitaire Diamond are known from Mr. Hideyoshi Fukazawa of the manufacturing distributor of FORTY-ONE.

Taro: He talled me that he had learnt FORTY-ONE from his grandfather.

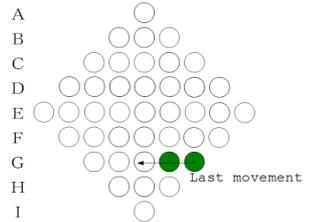
Movement of stone



Taro: In the following description, "The stone being removed at the beginning of the game" is said "First stone", and "The stone to be left at the last of the game" is said "Last stone".

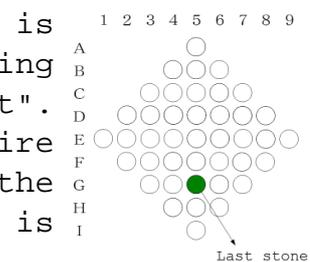
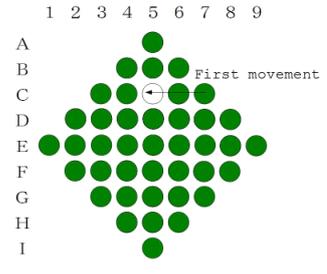
And "Only one stone is left at the

last of the game" is said "the game is ended".



Further, "Moving stone at first after removing First stone" is said "First movement", "Moving stone at second" is said "Second movement" and "Moving stone at last" is said "Last movement".

Hana: I think that Peg Solitaire Diamond is very difficult to end the game. Do you know how the game is



ended?

Taro: You should know some conditions necessary to end the game.

Hana: Necessary conditions?

Taro: For example, Do you know which stone is First stone?

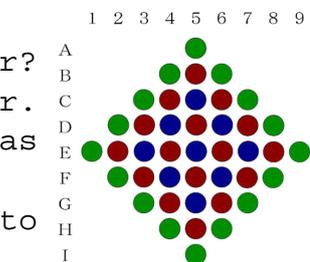
Hana: Is it necessary to know such a thing beforehand?

Taro: In Peg Solitaire Diamond, 41 stones are classified into three colored stone, e.g. 16 outside Green stones, 9 inside Blue stones, and 16 inside Red stones as right figure.

Hana: How do the three color stones differ each other?

Taro: Red stones are never moved to remove each other.

Taro: Green stones and Blue stones are also the same as Red stones.



Hana: Certainly, Green stones are moved into inside to become Blue stones, never Red stones.

Taro: Green stone become Blue stone to be moved into inside to remove Red stone.

Hana: Yet, why did you classify 41 stones into three colores?

Taro: Because we can easily consider the end of the game.

Hana: Do you think what relation between three colores and First stone exist?

Taro: All Green stones need to be moved for the game to be ended.

Hana: Why do you think so?

Taro: Because the game is never ended if even one of Green stones is not moved.

Hana: As for it, may not it be answered?

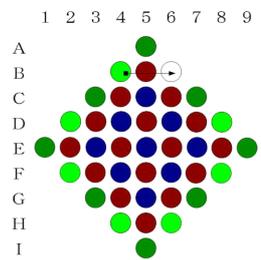
Taro: When paraphrasing it, because, when assuming that the game is ended, all Green stones are moved.

Hana: Do you say that the game is never ended if one Green stone remains?

Taro: Not so, one Green stone is moved as Last stone at the last of the game.

Hana: I cann't understand it well.

Taro: Though the reason will be explained later, Green stone might be moved from outside to outside only by First and/or Last movement of the game as right figure.



Taro: Therefore, then one Green stone is remained in outside hole B6 as Last stone at the last of game as left figure.

Hana: How actually is the Green stone moved?

Taro: For instance, by First movement after removing First stone B6, the Green stone B4 might be moved to hole B6 as shown in right above figure.

Hana: Yes, I see.

Taro: The Green stone B4 might be moved to hole B6 as Last stone by Last movement of the game.

Taro: In a word, all 16 Green stones are moved internally or outside in the game.

Hana: Is not Last stone remained outside?

Taro: Of course, Green stone might be moved from outside to outside to become Last stone by First and/or Last movement. In that case, Green stone is remained outside hole as Last stone.

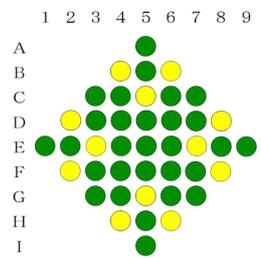
Hana: Indeed, is it so by the First and/or Last movement of the game?

Taro: However, if one of Green stones of right figure is removed as First stone, the game is never ended.

Hana: I have thought that First stone is good with all stones.

Taro: It is not so. In Peg Solitaire British, even if First stone is any stone, the game can be ended. But, in Peg Solitaire Diamond and French, First stone is not good with all stones.

Taro: First stone is either of Yellow stones of right above figure in Peg Solitaire Diamond.

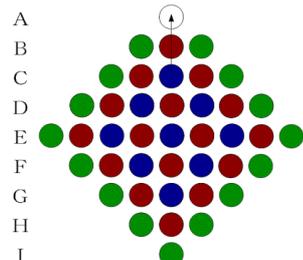


Hana: How does the game become if stone except Yellow stones is First stone?

Taro: The game is never ended.

Hana: However, how does the classification into three colors relate with this?

Taro: When Green stone A5 is First stone, Blue stone C5 is nothing but moved to hole A5 by First movement as shown in left figure.



Hana: At that time, Red stone B5 is removed by the movement of Blue stone C5.

Taro: As a result, the number of Red stones decreases by one to become 15 as shown in right chart, and then, the number of Green stone increases by one and becomes the former 16.

Taro: All 16 Green stones need to be moved necessarily for the game to be ended.

Taro: However, one Red stones is insufficient to move All 16 Green stones.

Taro: Therefore, one Green stone will be remained at last without being moved.

Hana: Is only one Green stone remained without being moved?

Taro: No, two stones or more in total are remained because the stone moved last remains inside beside the Green stone, and then the game is never ended.

Hana: Though the talk returns, there are only 12 Red stones that are adjacent to 16 Green stones. I am afraid that 4 Red stones are insufficient to move 16 Green stones.

Taro: Four Red stones D5, E4, E6 and F5 are moved to holes of Red stones which were removed by the movements of Green stones.

Hana: Indeed, it is so.

Taro: Green stones except First and Last stone are moved to inside to become Blue stones and Blue stones are removed by the movements of Red stones.

Hana: Why is Last stone excluded?

Taro: Because Last stone is not removed but moved and remained at last.

Hana: Then, do you say that Last stone is necessarily either of 16 outside Green stones.

Taro: Yes it is so in Peg Solitaire Diamond.

Hana: Is there no moved stone in Green stones?

Taro: No there is not. Because the game is never ended if one or more Green stones are not moved.

Taro: When Green stone is moved from outside to outside, there is the case when Green stone is remained outside as Last stone.

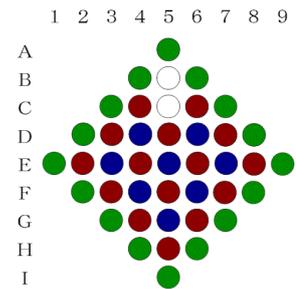
Hana: Why?

Taro: Though I think that I spoke the same thing to you many times, unmovable Green stones mean that at least one stone moved at last is remained too.

Taro: Then, two stones or more are remained necessarily. Therefore the game is never ended.

Hana: Why do you say so?

Taro: In Peg Solitaire Diamond, the number of stones are decreased by moving only Green stones and Red stones, and Last stone is left only by Last movement. The reason is that Peg Solitaire Diamond is such a game.



Hana: Though it is so.

Taro: Therefore, at least two stones are remained without fail. one is the stone moved at the last and the other is Green stone to be moved from outside to insaide or outside.

Hana: In a word, if two or more stones are remained, the game is not ended or cannot be continued.

Hana: Peg Solitaire Diamond is made well. The number of Green stones and Red stones is the same as 16 by chance.

Taro: I think that the same by chance oppositely makes easy to end the game of Peg Solitaire Diamond.

Hana: Do not you think it difficult?

Taro: The movements of stones are limited by this chance.

Taro: Green stones are remained outside without being moved if the number of Green stones is larger than the number of Red stones, and the game is never ended.

Hana: Such a game has no answer.

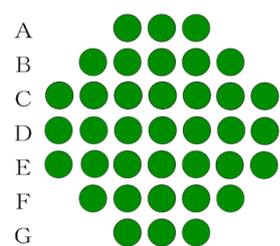
Hana: Do you know said such games actually?

Taro: That is Peg Solitaire Diamond 25 shown in right figure.

Hana: The number 12 of Green stones is certainly larger than the number 9 of Red stones.

Taro: Three green stones or more are necessarily remained outside if this Peg Solitaire Diamond 25 is

1 2 3 4 5 6 7 played.



Hana: Then, Solitaire 25 is not a game.

Taro: In opposite case, it becomes easy to end the game because Red stones too may be moved.

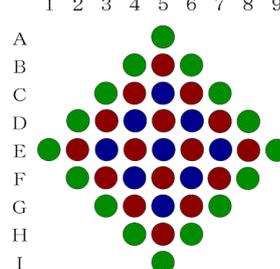
Hana: I think that the coincidence of the numbers of Green stones and Red stones makes Diamond interest and difficult.

Taro: It may be said so.

Hana: How do Peg Solitaire French and British become ?

Taro: limitations of stone movements in Peg Solitaire British and French are little compared with Diamond.

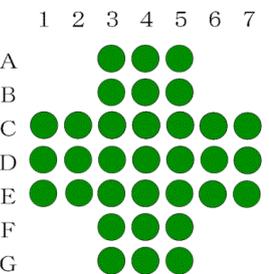
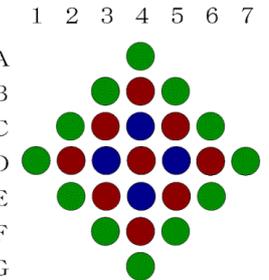
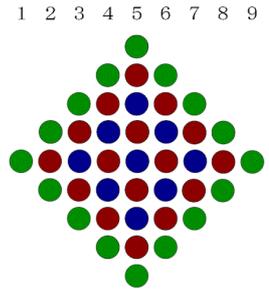
Taro: Because the degree of freedom of stone movements is less than Peg Solitaire British and French,



Diamond might be ended by trial and error.

Hana: The number of Green stones is always the same as the number of Red stones from First to Last movement in Diamond.

Taro: It is the same meaning as no movement of Blue stones.



Hana: Why do you say so?

Taro: Because Red stone is removed if Blue stone is moved.

Taro: As a result, the number of Red stones decreases to become less than the number of Green stones.

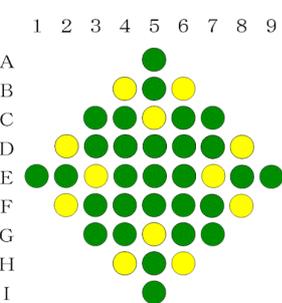
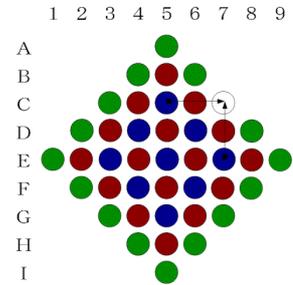
Hana: I have understood that Blue stones are never moved for the game of Diamond to be ended.

Taro: Don't forget it.

Hana: By the way, I have heard from you that there are the cases when Green stone is First stone for the game to be ended.

Taro: Surely, there are such cases.

Hana: When First stone is stone C7 as shown in left chart, Blue stone C5 or E7 is moved to hole C7 each removing Red stone C6 or D7. Then, the number 16 of Green stones becomes one more than the number 15 of Red stones. And the game is never ended.

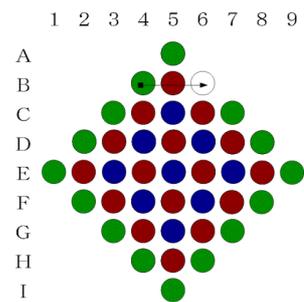


Taro: In the cases when First stone is stone A5 or C7, it is surely so. Therefore stones A5 or C7 and stones C3, E1, E9, G3, G7 and I5 symmetric to A5 or C7 never become First stone.

Hana: Then, which Green stones are First stone?

Taro: First stone is either of stone B6 or it's equivalent stones.

Hana: However, when stone B6 is First stone, by First movement, Blue stone D6 is moved to hole 6 and the number 16 of Green stones becomes one more than the number 15 of Red stones. Then the game is never ended.



Taro: In that case, since Green stone B4 is moved to hole B6 removing Red stone B5 by First movement after First stone B6. As a result, the number 15 of Green stones becomes the same as the number 15 of Red stones.

Hana: Indeed, it become surely so.

Taro: When either stone A5 or C3 is First stone, it certainly becomes so as you say.

Taro: But, as mentioned above, for instance, when the stone B6 is First stone, it is not necessarily so.

Hana: Green stone B4 is necessarily moved to hole B6 by First movement.

Taro: It has been shifted from "Movement of stone" to "First and Last stones".

Hana: Blue stone is never moved. Blue stone is necessary to be removed only by the movement of Red stone. And when moving, Green stone removes Red stone.

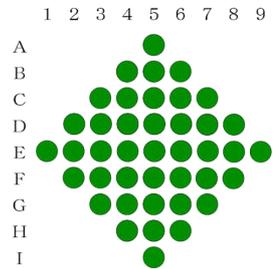
Taro: In short, Blue stone is never moved.

Taro: I think that it is all.

Taro: Let's examine "First stone" and "Last stone" at next chapter.

First stone

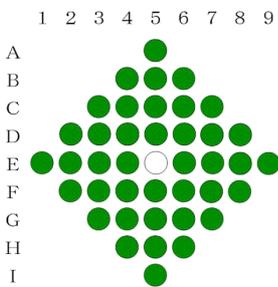
Taro: In the following description, "the stone being removed at the beginning of the game" is said "First stone", and "the stone being left at the last of the game" is said "Last stone". And "Moving stone at first after removing First stone" is said "First movement". And "Moving stone at second" is said "Second movement". And then, "Moving stone at the last of the game" is said "Last movement" corresponding to "39th movement".



Taro: Moreover, "Only one stone is left at the last of the game" is said "The game is ended".

Taro: First of all, let's think which of 41 stones is First stone.

Hana: Then, if First stone is removed by mistake, the game is never ended, do you say?



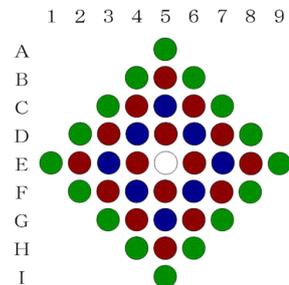
Taro: Yes it is. This Peg Solitaire Diamond is difficult because First stone need to be chosen from 41 stones.

Hana: Do you say that when First stone is chosen by mistake, the game is never ended.?

Taro: Yes it is so. However, even if any stone of Peg Solitaire British 33 is First stone, the game can be ended.

Hana: But, Peg Solitaire Diamond is not so, you say.

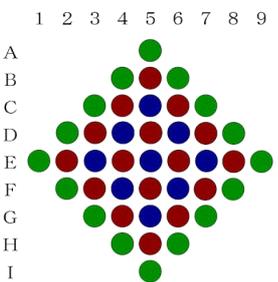
Taro: For instance, let us think about the case when Blue stone E5 as shown in right figure is First stone.



Hana: Why do you think such the case?

Taro: When this Blue stone E5 is First stone, the game is never ended.

Hana: Why?



Taro: Then, which stone do you move at first stone movement?

Hana: Either of Blue stones C5, E7, G5 and E3 need to be moved to hole E5.

Taro: At that time, Will you think that either of Red stones D5, E6, F5 and E4 is removed.

Hana: It is so as you say certainly.

Taro: However, if anyone of those Red stones is removed, the game is never ended.

Hana: Why can you decide that?

Taro: Inside 16 Red stones are necessary for outside 16 Green stones to be moved.

Hana: How does it relate to the end of the game?

Taro: If either of 16 Green stones is not moved to remain outside to the last, the game is never ended.

Hana: I can not understand the reason well.

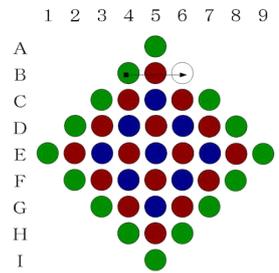
Taro: When either of 16 Red stones is First stone, either of outside 16 Green stones will remain outside without being moved.

Taro: Therefore, to end the game, 16 Red stones need to be removed only by the movement of outside 16 Green stones and then never become First stone.

Taro: And, the movement of outside 16 Green stones includes two stone movements from outside to inside and from outside to outside.

Hana: From outside to outside?

Taro: For instance, when Green stone B6 is First stone, Green stone B4 is moved to hole B6 as shown in right figure.



Hana: Indeed.

Taro: At that time, Red stone B5 is removed.

Taro: In a word, if either of Green stones is moved, Red stone adjacent to Green stone is removed.

Hana: How is it supplemented though it is thought that Four Red stones is insufficient?

Taro: However, four Red stones D5, E4, E6, and F5 are moved to holes of Red stones removed by Green stones during playing the game.

Hana: This Peg Solitaire Diamond is made well.

Taro: If one of Green stones remains outside without being moved, the game is never ended.

Hana: Does anyone of Red stones become Last stone?

Taro: No, I will explain in detail because it is important.

Taro: The stone that moves at the end of the game is Last stone.

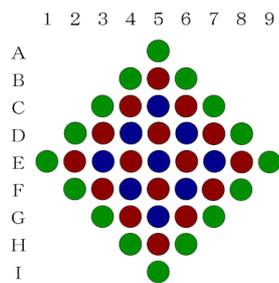
Taro: Then, at least two stones will remain, one is moved at last and the other is Green stone never moved.

Taro: In a word, because at least two stones remain, the game can not be continued. That is, the game doesn't end.

Hana: Indeed it is so. And, I could understand well that if either of four Red stones around the center is First stone, the game is never ended.

Taro: Though I think that you have already noticed, the number of Green stones and the number of Red stones need to be always the same to end the game.

Hana: It is why?



Taro: It is necessary to remove Red stones only by the movement of Green stone.

Taro: The stones moved during the game are Green stones and Red stones only.

Taro: In a word, Blue stones are never moved and only removed.

Taro: Green stone is moved to inside to become Blue stone. And at First or Last movement, Green stone may be moved from outside to outside.

Hana: I think that the same thing occurs when Blue stone is moved or when Red stone is First stone.

Taro: It is natural.

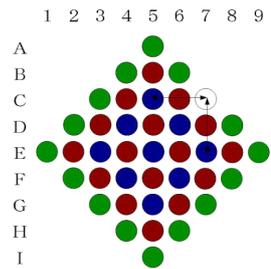
Hana: For example, Can Green stone C7 become First stone.

Hana: Either of Blue stones C5 and E7 is moved to hole C7 as shown in right figure.

Taro: At that time, either of Red stone C6 or D7 is removed.

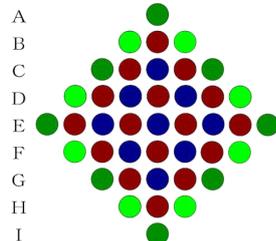
Hana: The number of Green stones becomes 16, and the number of Red stones becomes 15.

Taro: It is the same in the case when Green stone A5 is First stone.



1 2 3 4 5 6 7 8 9

Hana: Does Green stone never become First stone?



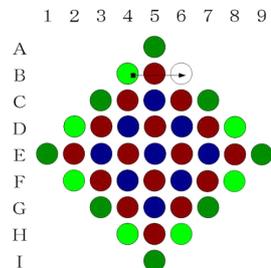
Taro: It is not always so.

Hana: Does it mean that there is the case when Green stone is First stone?

Taro: Let's explain about the case.

Taro: When either of eight bright Green stones B4, B6, D2, D8, F2, F8, H4

and H6 as shown in left figure is First stone, the number of Red stones and the number of Green stones become the same by First movement of the stone.



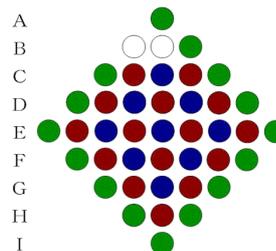
Hana: How is First movement executed?

Taro: For instance, when bright Green stone B6 is First stone, bright Green stone B4 is moved to hole B6 by First movement as shown in right figure.

Hana: Is it the same for other bright Green stones?

1 2 3 4 5 6 7 8 9

Taro: The same thing is applied for other seven Green stones because they are symmetrical to bright Green stone B4.



Hana: What is symmetry?

Taro: A original figure A (an array or a pattern is acceptable) is called symmetric when the original figure A is moved (for instance, by rotation or by

return) to become a figure A' and when it is possible to overlap the figure A' with the original figure A.

Taro: When rotating or returning the original array of bright Green stones, the rotated or returned array is able to be overlapped to the original array.

Hana: I was able to understand really well.

Taro: Then, when Green stone B4 is moved to hole B6 by First movement, Red stone B5 is removed.

Hana: Certainly, the number of Green stones and the number of Red stones become the same 15.

Taro: Therefore, it remains a possibility that the game may be ended in the case when Green stone B6 is First stone.

Hana: I thought only Blue stone D6 was moved to B6.

Taro: In a word, there is a possibility that the game can be ended in the case when either of eight bright Green stones is First stone.

Hana: Only the possibility?

Taro: Actually the game can be ended when either of bright Green stones is First stone.

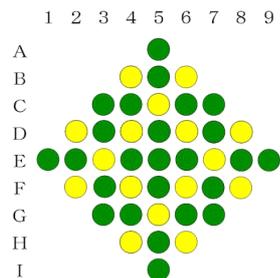
Taro: However, it is necessary to advance the game not to move Blue stones from First to Last movement.

Taro: However, the stone movements from First to Last by which the game end without fail are not understood yet.

Hana: Then, do you say that the game might not end even if not making a mistake on the way?

Taro: It is so. Yet it has not been discovered how to advance the game of Solitaire 41 to end without fail.

Taro: The stone movements of tens of 100 kinds to end the game are understood from using the personal computer.



the game are understood from using the personal computer.

Taro: And, the personal computer only displays the stone movements to end the game, and doesn't answer of why.

Hana: Whichever of nine Blue stones is First stone?

Taro: It is not so. Have you already forgotten?

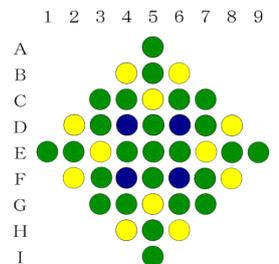
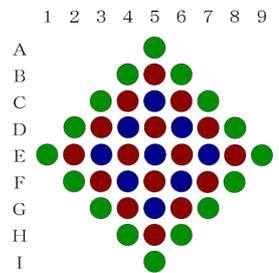
Taro: When Blue stone E5 is First stone, the game is never ended.

Hana: Then, First stone is which stone?

Taro: After all, First stone becomes either of 16 Yellow stones shown in left above figure.

Taro: However, when either of Blue stones D4, D6, F4, and F6 (right figure) is First stone, it is understood not to be able to end the game.

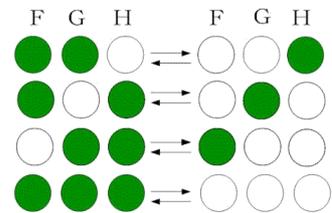
Hana: Why do?



Taro: The reason is described in "Charosh's transformations"(38-40 page) of "II Solitaire (Wisdom Sesame Puzzle)" of "Mathematical Game II" published by KOUDANSHA on April 1, 1991, written by Martin Gardner, translated by Mr. Shigeo Takagi. www.cst.cmich.edu/users/grahalsw/

Taro: This transformations will be explained in detail in "Virtual rule"

Taro: Moreover, this is introduced as "transformations gave by Suremain de Missery"(P. 196-197) in "The Ins & Outs of Peg Solitaire" wrote by Mr. John D. Beasley of Oxford University in Britain published in 1985.



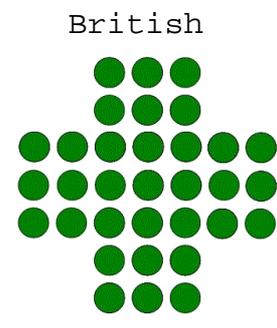
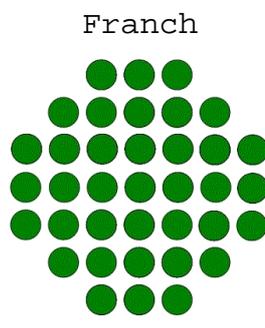
Hana: How was it written?

Taro: It is written that it is possible to judge the end of Peg Solitaire game by using the transformations, even if the solitaire has any kind of array of stones.

Hana: What are the transformations?

Taro: Three stones arrayed vertically or horizontally are removed.

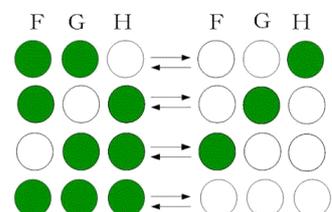
Taro: Zero, one or two stones remain at the last if the stones are removed by using the transformations.



Hana: How do you do when three stones do not array horizontally or vertically?

Hana: And how do you do when there is no stone in the middle or when there is no stone in both sides?

Taro: The number of stones is decreased by using transformations as shown in right figure.



Hana: Why are such transformations used?

Taro: The purpose is to relax Usual rule.

Hana: What is the purpose to relax Usual rule?

Taro: The purpose is to enable stone movements which are impossible by Usual rule.

Hana: Please explain the relaxed rule concretely.

Taro: From now on, I decided to call the relaxed rule "Virtual rule".

Hana: Why do you change the relaxed rule to "Virtual rule"?

Taro: Because we cannot play the game by "Virtual rule" in usual Peg Solitaire board.

Hana: I understood that the game may be virtually played by "Virtual rule" in our brains.

Taro: And in usual board, only one stone can be put in one hole. However, in Virtual rule, more than two stones are already in one hole .

Taro: Further in Virtual rule, hole with even number stones is the same as with no stone, and hole with odd number stones is the same as with one stone.

Taro: Moreover, if a hole with no stone is stepped over, the hole is changed with one stone.

Taro: Therefore, it is possible to move one stone from hole with no stone to hole with one stone stepping over hole with no stone.

Taro: As a result, hole with no stone is changed with one stone, and hole with one stone is changed with no stone.

Hana: Is anything possible?

Taro: Everything possible in Usual rule is possible in "Virtual rule".

Hana: Cannot the game be executed in usual Peg Solitaire board by Virtual rule?

Taro: It is so, but as explained already, the game may be played virtually in the brain by Virtual rule.

Hana: I think that Virtual rule may not have any impossibility.

Taro: No, there is the case the game can not be ended even by Virtual rule.

Hana: For instance, what is the case?

Taro: If either of four Blue stones is taken at first as shown in right figure, the game cannot be ended even by Virtual rule.

Hana: Is it true?

Taro: It is surely so. There are cases when the game cannot be ended because two stones remain even by Virtual rule.

Hana: I see. If a game is impossible by Virtual rule, the game is impossible by Usual rule.

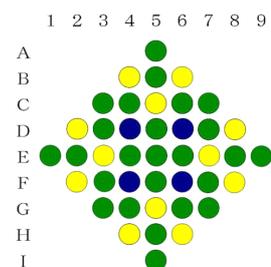
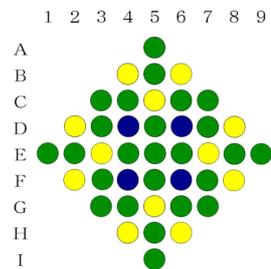
Taro: You are very sharp.

Hana: I am glad.

Taro: However, some games can not be ended by Usual rule even if the game can be ended by Virtual rule.

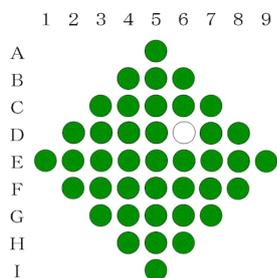
Hana: What do you mean by said games?

Taro: When a game can be ended by Virtual rule, the game has only possibility to be ended by Usual rule.



Hana: Do you mean that the game has only possibility to be ended, but not possible by all means?

Taro: Yes!



Hana: I see. It is a required condition, but it is not a sufficient condition.

Taro: In other words, it is impossible by Usual rule in any case when it is impossible by Virtual rule.

Taro: It is sometimes impossible by Usual rule even if possible by Virtual rule.

Hana: That the game can be ended by Virtual rule means only possibility that the game may be ended by Usual rule.

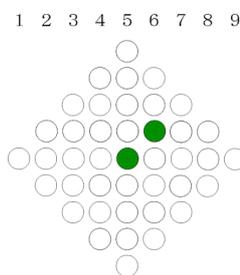
Taro: Therefore, returning to the original problem, let us try to think about the case when Green stone D6 is removed as First stone (refere to left above figure).

Hana: By the way, what is the purpose to play the game by the Virtual rule?

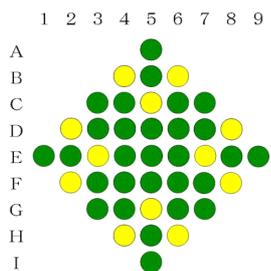
Taro: The purpose of Virtual rule is to clarify possibility or impossibility of the end of the game by Usual rule.

Hana: First of all, let us play the game by Virtual rule.

Taro: Then, when Peg Solitaire Diamond is played by Virtual rule, two Green stones D6 and E5 are left as shown in right figure and the game cannot be continued further.



Hana: Two Green stones D6 and E5 are never transformed into one stone by Virtual rule.



Taro: In other words, even by Virtual rule, the game cannot be ended when stone D6 is First stone.

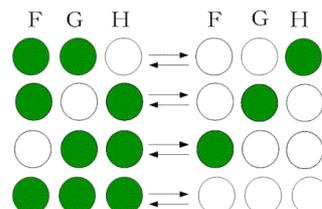
Hana: Thus either of Blue stones D4, D6, F4 and F6 can not be First stone.

Taro: After all, First stone is either of 12 Yellow stones as shown in left figure because stones D4, D6, F4 and F6 are excluded.

Hana: What is more different than Charosh or Missery transformations as for Virtual rule?

Taro: You asked me very good question.

Taro: Virtual rule was created first by Mr. Hiroki Hayashi (patent attorney). To play the game by using Charosh or Missery transformations, he changed Usual rule to Virtual rule.



Hana: What is different between Charosh or Missery transformations and Virtual rule ?

Taro: Virtual rule is different from Usual rule in the following two points.

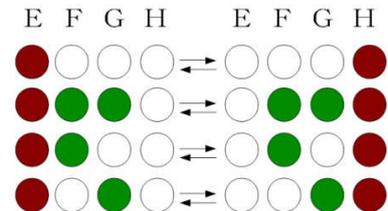
Taro: At first, two or more stones may exist in one hole.

Taro: At second, when the number of stones in a hole is even, the hole is expressed by "nothing" or "no stone", but when the number of stones in a hole is odd, the hole is expressed by "existence" or "one stone".

Taro: Additionally, the others are the same as Usual rule.

Taro: Also, the transformations shown in right figure are possible by Virtual rule.

Taro: As understood from right figure, Red stone in hole E can be moved to hole H without changing the array of stones of holes F and G. And the reverse is similar, too.



Hana: Surely Red stone in hole E is moved to hole H.

Taro: This Red stone movement by Virtual rule is quite different from Charosh or Misery transformations.

Hana: It is so.

Taro: Hereafter, such Red stone movement by Virtual rule will be called "transfer" to distinguish from stone movement by Usual rule.

Hana: What convenient is Virtual rule!

Taro: The rule like Virtual rule is used in computer operation.

Hana: How is it operated?

Taro: It is called EXOR (Exclusive OR).

Hana: Though I know OR Boolean operations.

Taro: OR Boolean operations are as follows.

$$1+1=1 \quad 1+0=1 \quad 0+1=1 \quad 0+0=0$$

Hana: What operation of EXOR is different from OR?

Taro: To be EXOR Boolean operation is $1+1=0$.

Taro: "EX" of EXOR expresses "exclusive".

Hana: I see. When "1" is assumed "one stone" and "0" is assumed "no stone", "one stone" + "one stone" become "no stone".

Taro: By Virtual rule, it can be interpreted that there are even number stones in hole with "no stone" and odd number stones in hole with "one stone".

Hana: They are different from EXOR Boolean operation.

Taro: Only similar.

Hana: Why do you play the game by Virtual rule for?

Taro: We may judge by Virtual rule whether the game has possibility to be ended.

Taro: Further, by Virtual rule, we can predict First stone and Last stone.

Hana: How can you predict them? Teach me it!

Taro: Do not you show me it?

Hana: Teach it early without being pretentious !

Taro: We can play the game without removing First stone by Virtual rule.

Hana: That is true! Because stones more than two have been already in each hole.

Taro: Then let us play the game.

Hana: Each hole has odd number stones because the each hole is "one stone" at the beginning of the game.

Taro: Let us change the number of stones of each hole from odd to even, namely, from "one stone" to "no stone".

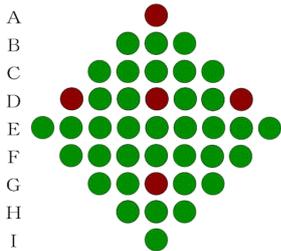
Hana: Should we reduce the number of holes with "one stone" as much as possible by Virtual rule?

Taro: It is so.

Hana: It is troublesome comparatively. Don't you know the simple expedient more?

Taro: To tell the truth, I know the simple expedient more.

1 2 3 4 5 6 7 8 9



Hana: Please teach it from the beginning.

Taro: You can not understand the value of the simple expedient because you don't experience troublesome works to be performed without using the simple expedient.

Hana: Because you do not teach it, I cannot understand the value.

Taro: Anyway, Five Red stones shown in left figure are equivalent stones to be transfered each other according to Virtual rule.

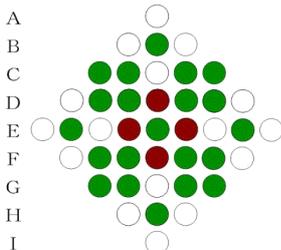
Hana: It is so.

Taro: Therefore, four Red stones A5, D2, D8 and G5 are transfered to hole D5.

Hana: Then holes A5, D2, D8 and G5 change from "one stone" to "no stone".

Taro: How does hole D5 turn out?

1 2 3 4 5 6 7 8 9



Hana: I think that there are five stones in hole D5.

Taro: I think that it is not so.

Hana: Why is it?

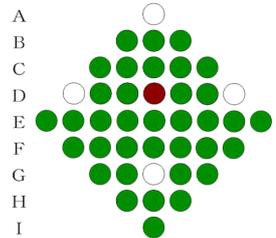
Taro: By Virtual rule, because the number 5 is odd, Red stone D5 is "one stone" as shown in right figure.

Hana: It was so.

Taro: We can say the same thing for holes E4 and E6 and F5.

Hana: Then, as shown in left figure, holes E4 and E6 and F5 become "one

1 2 3 4 5 6 7 8 9



stone" and holes equivalent to holes E4 and E6 and F5 become "no stone".

Taro: It is so.

Hana: What do you do for the next?

Taro: You should think by yourself occasionally.

Hana: So often. This time, Blue stone D4 is sure to be examined.

Taro: Blue stones D7, G4 and G7 are transferred to D4 and then holes D7, G4, G7 and D4 become "no stone".

Hana: Why does hole D4 too become "no stone"?

Taro: Because there are four stones of the even number in hole D4.

Hana: It is so.

Taro: The same thing is applied to stones D6, F4 and F6, because they are symmetric to stone D4.

Hana: Then, holes D6, F4 and F6 symmetric to hole D4 become "no stone" as well as hole D4.

Taro: It is so.

Hana: However, holes with "one stone" are disappeared fairly.

Taro: Don't you think something from left figure?

Hana: Five Green stones are mutually equivalent stones.

Taro: Therefore, as shown in right figure, since those Green stones are transferred to hole E5, hole E5 becomes "one stone", and holes B5, E2, E8 and H5 become "no stone".

Hana: I think that it is impossible to reduce holes of "one stone" any further.

Taro: It is not so.

Taro: Because Red stone E4, Green stone E5, and Red stone E6 are arrayed horizontally, holes E4, E5 and E6 are converted into "no stone".

Hana: It was so. I have forgot.

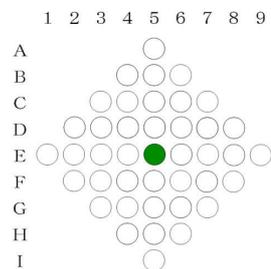
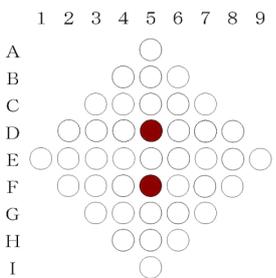
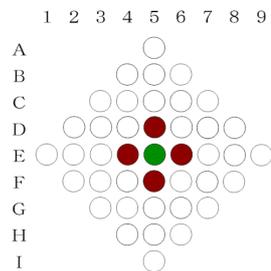
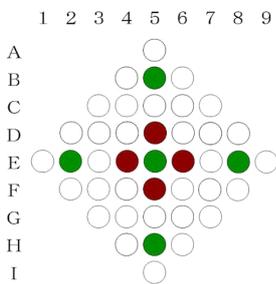
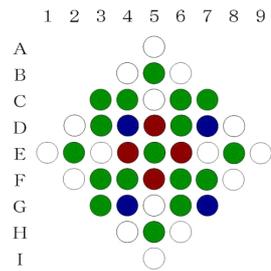
Taro: And, two Red stones D5 and F5 are converted into Green stone E5 with "one stone".

Hana: After all, in Peg Solitaire Diamond, when the game was played without removing First stone by Virtual rule, stone E5 only remains.

Taro: To be precise, also the only one stone may remain in either of holes E5, B5, E2, E8 and H5. Because stones B5, E2, E8, H5 and E5 are equivalent each other.

Hana: However, is it strange?

Taro: If First stone is stone E5, all stones are disappeared.

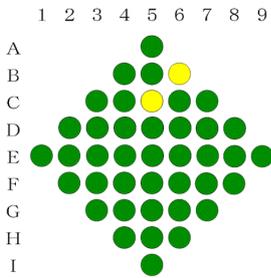


Taro: Can you explain what it mean?

Taro: Since Last stone could not exist, the game is never ended.

Hana: Do you think what it means by Usual rule?

Taro: It means that it is absolutely impossible to end the game by Usual rule because it is impossible even by Virtual rule.



Hana: In other words, if stone E5 is First stone, the game is never ended by Usual rule.

Taro: It is so.

Hana: However, I have heard from you that when Yellow stone B6 or C5 is First stone, there is a possibility to end the game.

Taro: You asked me nice question.

Hana: Is it possible even by Virtual rule?

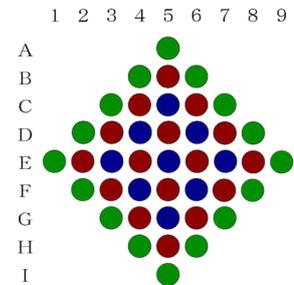
Taro: Before answering the question, I should explain the reason why stone E5 only remains when the game was played without removing First stone by Virtual rule.

Hana: You have explained the reason enough before.

Taro: No, I am thinking about applying the reason to explain other Solitaires British, French and Triangle, etc.

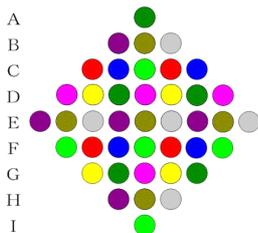
Hana: What is the reason you apply Virtual rule to them?

Taro: The stones of Peg Solitaire Diamond has been considered up to now by classifying into three kinds (Red stones, Green stones and Blue stones) as shown in right above figure.



Hana: It so.

Taro: However, the forty one stones of Peg Solitaire Diamond are classified into nine different colors by Virtual rule as shown in left figure.



Hana: What will you do by classifying the forty one stones into nine colors?

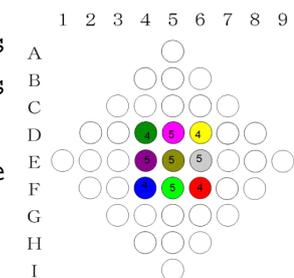
Taro: The same color stones can be transfered to one hole because they are equivalent each other as shown in left figure.

Hana: Is not three colors?

Taro: It is only so because the array of Solitaires Diamond and French 37 and British 33 stones is symmetry.

Hana: In a word, stones of any Peg Solitaire are classified into nine different color stones.

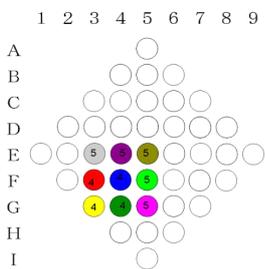
Taro: Nine different color stones D4, D5, D6, E4, E5, E6, F4, F5 and F6 are never transfered mutually.



Hana: Actually, even by Virtual rule, nine different color stones of

Green , Magenta, Yellow, Red, Brown, Ash, Bright Blue, Bright Green and Bright Red cannot be transferred mutually.

Taro: By Virtual rule, no matter how the stones of Solitaire is arrayed, each stone is equivalent to either of this nine different color stones.

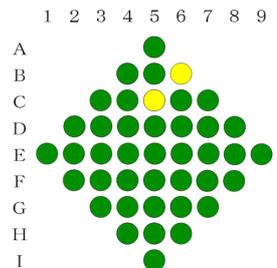


Taro: Therefore, all stones are transferred to either of holes D4, D5, D6, E4, E5, E6, F4, F5 and F6.

Hana: I could understand well the reason why Virtual rule can be applied to Peg Solitaire of what kind of array.

Taro: Nine different color stones may not be arrayed around the center as shown in left figure.

Hana: Indeed, the number of each different color stones is countered in the array. And if the number is odd, its' color stone becomes "one stone". And if the number is even, its' color stone becomes "no stone".



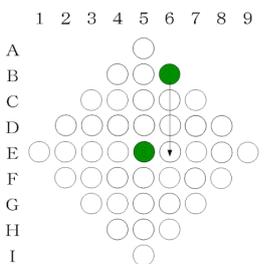
Taro: Therefore, you may immediately understand from left above figure that stone E5 only remains.

Hana: By Virtual rule, it is possible to arrange the array of stones easily by transferring the stones.

Taro: Because it is possible to move each color stone without changing the array of other color stones.

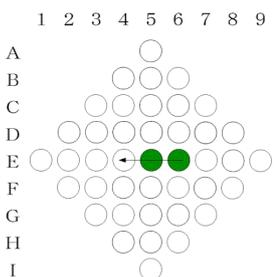
Hana: It becomes very easy to arrange the array of stones.

Taro: In mathematics and physics, it is said that Superposition principle might be applied.



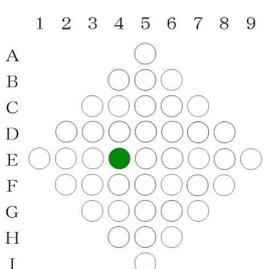
Hana: However, what relation exist between arranging of the array of stones and making Yellow stone B6 or C5 First stone ?

Taro: By Virtual rule, for instance, when stone B6 is First stone, hole B6 become "one stone" as shown in left figure.



Hana: It is certainly so.

Taro: Then, stone B6 is transfered to hole E6.

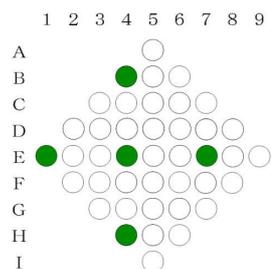


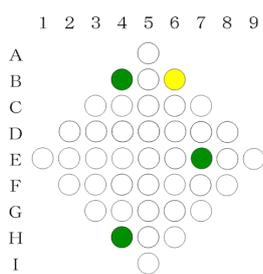
Hana: However, two stones remain still.

Taro: In addition, those two stones are made one stone E4 by conversion as shown in upper figure.

Hana: But, according to Usual rule, Last stone can not be there.

Taro: Then, there is a possibility that Last stone remains in either of stones B4, E1, E7





and H4 because those stones B4, E1, E7 and H4 are equivalent to the stone E6.

Hana: Certainly, it is possible that Last stone remains in either of holes B4, E7 and H4 by Usual rule.

Taro: It has become the story of Last stone.

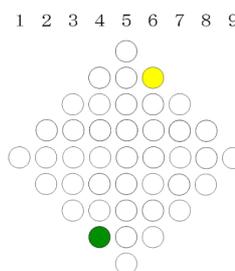
Hana: Then, shall we examine Last stone?

Taro: However, the game of Last stone H4 by First stone B6 cannot be ended by Usual rule.

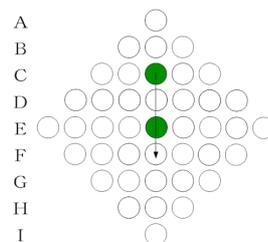
Taro: It is actually impossible though it is thought that there is a possibility to end the game even by Usual rule.

Taro: "Last stone H4 by First stone B6" will be proven to be impossible in another paragraph.

Hana: Please explain easily the case when stone C5 is First stone.



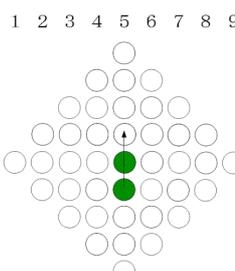
Taro: When stone C5 is First stone, two stones C5 and E5 remain by Virtual rule as shown in left figure.



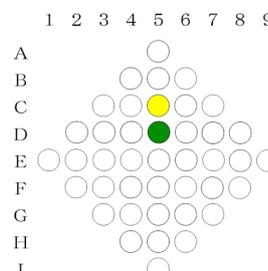
Taro: Then, stone C5 is transferred to hole F5.

Hana: Because indeed, stone C5 is equivalent to stone F5.

Taro: In addition, the transferred stone F5 is moved to hole D5 to remove

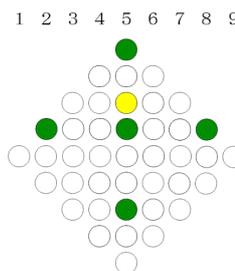


Green stone E5.



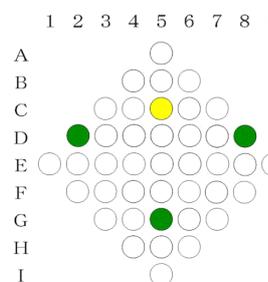
Hana: As a result, Green stone D5 remains as Last stone as shown in left figure.

Taro: there is a possibility that Last stone may be also either of Green stones A5, D2, D8 and G5 because these stones are equivalent to Green stone



D5.

Hana: However, Last stone could not remain in holes A5 and D5 by Usual rule.

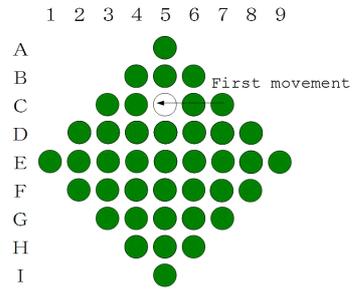


Taro: Therefore, there is a possibility by Usual rule that Last stone is either of Green stones D2, D8 and G5 when Yellow stone C5 is First stone as shown in left figure.

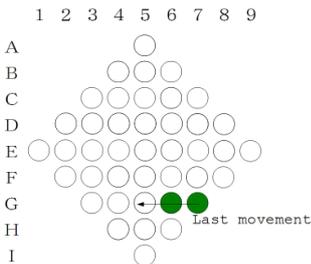
Last stone



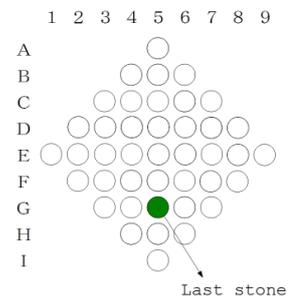
Taro: In the following description, "Stone removed at the beginning of the game" is said "First stone", and "Only one stone left at the end of the game" is said "Last stone". And "Moving stone at first after



removing First stone" is said "First movement". And "Moving stone at second" is said "Second movement".

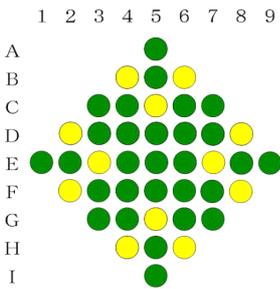


And then, "Moving stone at the last of the game" is said "Last movement" corresponding to "39th movement".



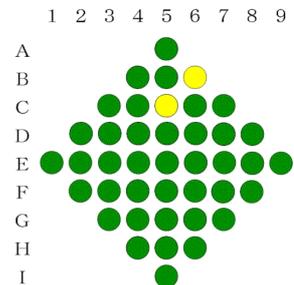
Taro: Moreover, "Only one stone is left at the last of the game" is said "The game is ended".

Hana: Last stone is predictable from First stone by Virtual rule.



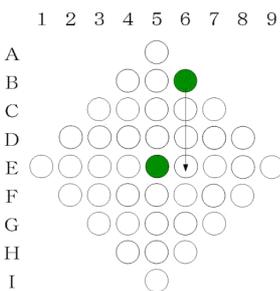
Taro: It is so.

Taro: First stone was either of twelve Yellow stones as shown in left figure.



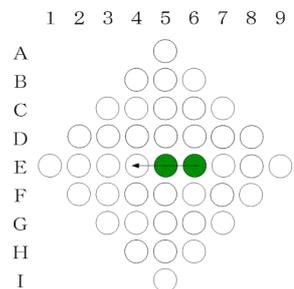
Hana: It was so.

Taro: Let's examine Last stones in the case when Stones B4 and C5 are First stone.

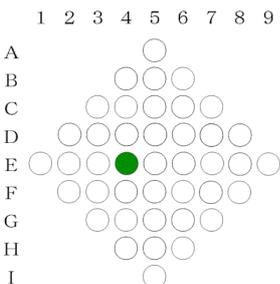


Taro: Because either of the other Yellow stones is symmetric with either of Stone B4 or C5, they need not to be examined.

Taro: Then let us examine about Last stone when First stone is Yellow stone B6.



Taro: By Virtual rule, two Stones B6 and E5 remain as shown in left chart.

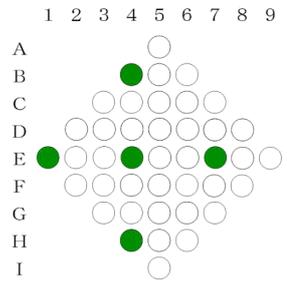


Taro: And, Stone B6 is transferred to Hole E6 as shown in left chart, because Stone B6 is equivalent to Stone E6.

Taro: In addition, when Stone E6 is moved to Hole E4, Stone E5 is removed as shown in the right chart, and Stone E4 only remains as shown in the left chart.

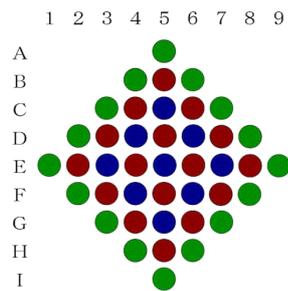
Hana: In a word, when Stone B4 is First stone, Last stone remains in Hole E4 by Virtual rule. In other words, Stone E4 is Last stone.

Taro: No, either of Green stones B4, E1, E7 or H4 equivalent to Stone E4 as shown in right figure has a possibility to become Last stone, too.



Hana: Surely by Virtual rule, stones equivalent to Stone E4 may become Last stone, too.

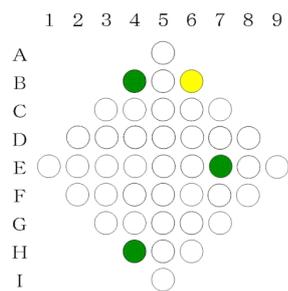
Taro: However, I think that Last stone can not be remained in Hole E4 by Usual rule.



Taro: How wise you are! By Usual rule, Stone of Last movement is either of Green stones as shown in left figure which cannot become Stone E4. Therefore, Last stone is never Stone E4.

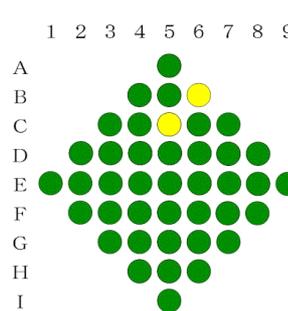
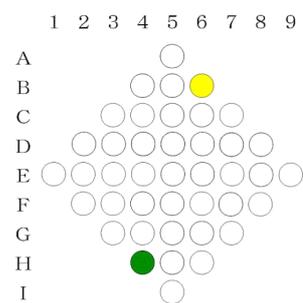
Taro: Either of Stones A6, E3 and H6 which may become Last stone by both of Usual and Virtual rule actually become Last stone.

Hana: In a word, either of Green stones B4, E7 and H4 may become Last stone when Stone B6 is First stone.



Taro: Though repeats many times, Last stones predicted by Virtual rule has only a possibility, but not necessarily to become Last stone by Usual rule.

Taro: Though Stone H4 has a possibility to become Last stone when Stone B6 is First stone, but actually Stone H4 can not become Last stone.



Hana: Stone H4 cannot actually become Last stone though Stone H4 has a possibility of Last stone by both of Usual and Virtual rule.

Taro: This is just the example which has a possibility but actually impossible.

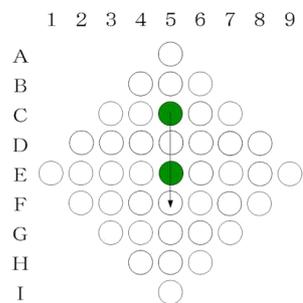
Hana: Predictions by both of Usual and Virtual rule are not sufficient

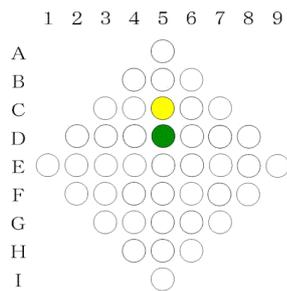
but necessary conditions.

Taro: It is possible to say like that.

Hana: Next, when Stone C5 is First stone, which stone becomes Last stone?

Taro: By Virtual rule, two Stones C5 and E5 are remained as shown in

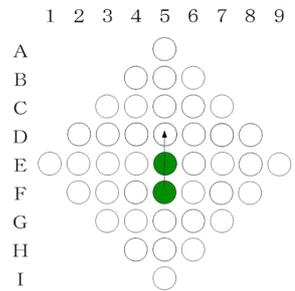




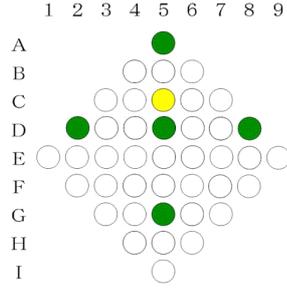
right upper chart.

Taro: And, Stone C5 is transferred to hole F5 as shown in left figure because Stone C5 is equivalent to Stone F5.

Taro: In addition, when Stone F5 is moved to hole D5, Stone E5 is removed and then only Stone D5 is remained as

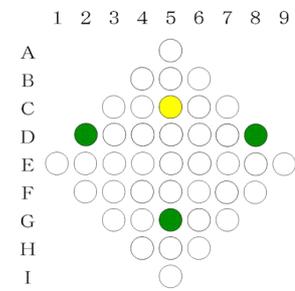


Last stone as shown in right figure.



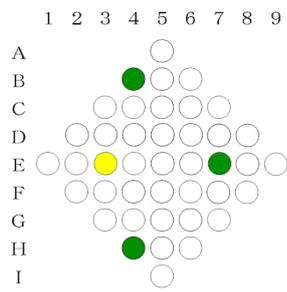
Taro: Therefore, by Virtual rule, when Stone C5 is First stone, Last stone may be too either of Stones A5, D2, D8 and G5 equivalent to Stone D5 as shown in left figure.

Taro: By Usual rule, because Last stone can't be either of Stones A5 and D5, Last stone is either of Stones D2, D8 and



G5.

Hana: By Usual rule, when Yellow stone C5 is First stone, either of Green stone D2, D8 and G5 may become Last stone.



Taro: It is so.

Hana: I think that the same thing can be said about Stone E3 which is symmetrical to Yellow Stone C5.

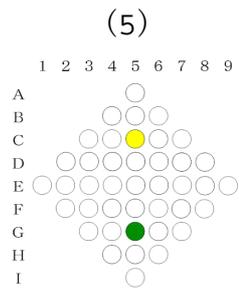
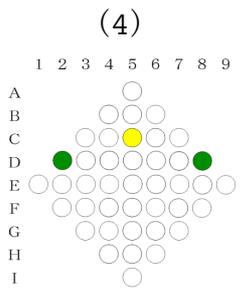
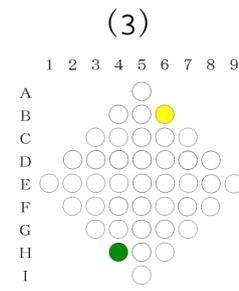
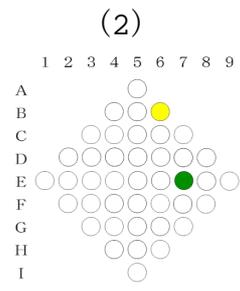
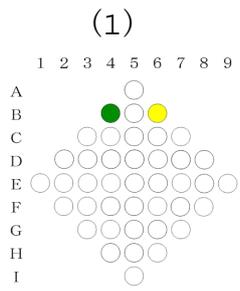
Taro: Of course, it is so. About Stones G5 and E7, the same thing can be said too.

Taro: When Yellow stone E3 is First stone, either of Green stones B4, D7 and H4 may become Last stone .

Hana: Oh! Symmetry

Taro: There are five following combinations (1)-(5).

- (1) First stone B6 and Last stone B4
- (2) First stone B6 and Last stone E4
- (3) First stone B6 and Last stone H4
- (4) First stone C5 and Last stone D2 or D8
- (5) First stone C5 and Last stone G5



Hana: Indeed, Last stones are predictable from First stones.

Taro: But they have only a possibility not a necessity.

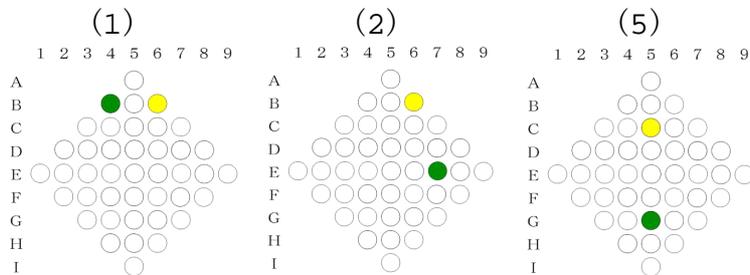
Taro: And the case (3) "First stone B6 and Last stone H4" is actually impossible.

Taro: The other cases (1), (2), (4) and (5) are actually possible.

Taro: The case (2) is only different from the case (4) at the point where Last stone is replaced with First stone. And according to Leibnitz, the game can be ended even if Last stone is replaced with First stone each other. Then, the case (2) is substantially the same as the case (4).

Hana: After all, the combinations of First stone and Last stone are following three cases (1), (2) and (5) as shown in under figures, don't you?

Taro: That is all right.



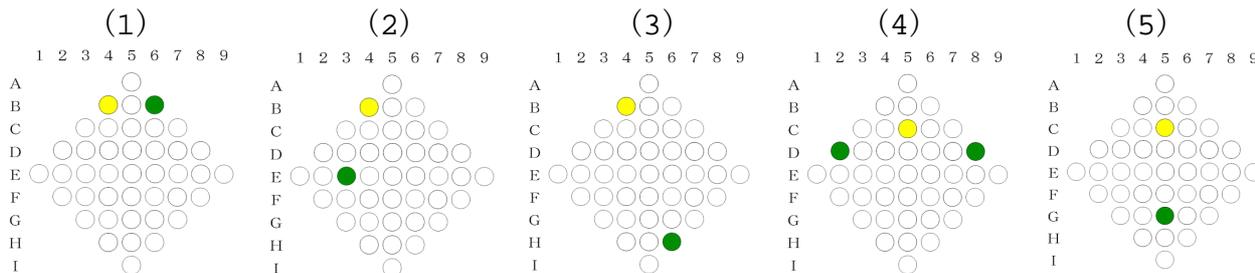
Opposite procedure

Hana: There are only five kinds of combinations (1)-(5) of First stone and Last stone that can end the game by Virtual rule.

Taro: It is so.

Hana: It is unexpectedly few.

Taro: The same one and the substantially impossible one are included in these five kinds of combinations (1)-(5).



Hana: Which?

Taro: When First stone is Yellow stone B4 as right figure, it is actually impossible by Usual rule though there is the possibility that Last stone remains in hole H6.

Taro: It is impossible from the analysis of the mobility pattern of stones though it is not possible to explain theoretically.

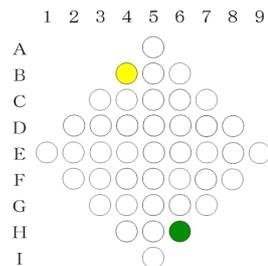
Hana: What is the reason?

Taro: When concretely looking for the procedure for ending the game actually, the reason will be proven later.

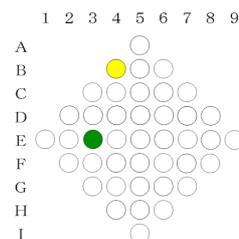
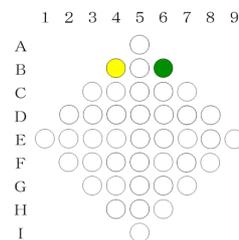
Taro: Anyway, we will think by excluding it for Combination (3) now.

Taro: And, First stone has only changed places into Last stone with Combinations (2) and (4).

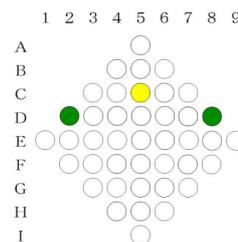
Combination (1): First stone B6 and Last stone B6



Combination (2): First stone B4 and Last stone E3



Combination (4): First stone C5 and Last stone D2 or D8



Hana: Why are Combination (2) and (4) the same? Are not you quite differing First stone and Last stone?

Taro: Combination (2) comes in succession with Combination (4) if Combination (2) is rotated to left by 90 degrees after turning of right and left.

Hana: Still, the color is different.

Taro: It is because Combinations (2) and (4) are mutually in the relation between the table and the back.

Hana: What meaning is relation between table and back?

Taro: In Combination (2), when First stone is removed from hole D2, Last stone remains in hole C5.

Taro: Last stone remains in hole D2 if First stone is removed from hole C5 and the game is played by completely opposite procedure.

Hana: The reverse procedure?

Taro: The order ($g \rightarrow h, e \rightarrow f, c \rightarrow d, \text{ and } a \rightarrow b$) opposite to the order ($a \rightarrow b, c \rightarrow d, e \rightarrow f, \text{ and } g \rightarrow h$).

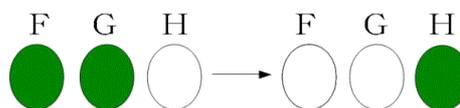
Hana: Then, what is Leibnitz's reverse procedure?

Taro: Leibnitz's reverse procedure is the order ($h \rightarrow g, f \rightarrow e, d \rightarrow c, b \rightarrow a$) reverse to the order ($a \rightarrow b, c \rightarrow d, e \rightarrow f, \text{ and } g \rightarrow h$).

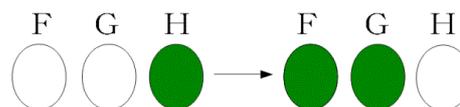
Hana: Both are different though they look like certainly well.

Taro: In the letter of 1710, Leibnitz has described the enjoyment of Peg Solitaire by repeating the procedure of "When Last stone H returns to hole F stepping over hole G, the stone is generated in hole G".

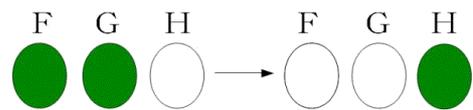
In the opposite procedure of Usual rule, stone F moves to hole H removing stone G.



In the Leibnitz's reverse procedure, stone H returns to hole F generating one stone in hole G.



In Leibnitz 's reverse procedure, if the presence of the stone is replaced, it becomes the same as the opposite procedure of Usual rule.

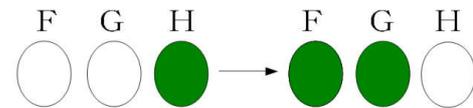


Hana: Leibnitz's reverse procedure is the same as turning the cinefilm oppositely only by time's going back.

Taro: However, an important hint is included in Leibnitz's reverse procedure.

Hana: Hint?

Taro: In Leibnitz's reverse procedur, Last stone H returns to hole F stepping over hole G and generating one stone in hole G.



Hana: It so.

Taro: However, it is not so when modifying one's views.

Hana: How do you modify your views?

Taro: It is thought that hole F moves to stone H stepping over hole G, generating one stone in hole G and removing stone H".

Hana: It is unexpected.

Taro: Only the presence of the stone reverses, and any rules are not changed.

Hana: It is certainly so when pointed out like that.

Taro: However, stone H is Last stone in Leibnitz's reverse procedure, but stone H is First stone in opposite procedure of Usual rule.

Hana: In a word, when the presence of the stone is reversed and when First stone is replaced with Last stone, Leibnitz's reverse procedure is the same as opposite procedure of Usual rule.

Taro: Leibnitz is great.

Hana: Then, the game ends at First stone if playing the game from Last stone.

Taro: It is so.

Hana: The story that consents strangely though it is natural.

Taro: Therefore, if the presence of the stone is reversed and if First stone is replaced with Last stone, Combination (2) becomes Combination (4).

Hana: It is said, "Opposite of Combination (2) is Combination 4".

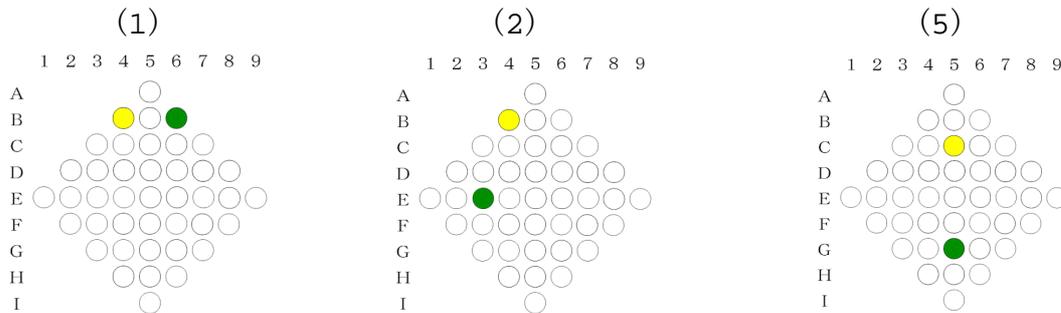
Taro: Combination 2 and Combination (4) are in the relation of the inside and outside.

Hana: It was so.

Taro: if the procedure of Combination (2) is reversed, it becomes the procedure of Combination 4.

Taro: When if the procedure of Combination (4) is reversed, it becomes the procedure of Combination (2) though nature.

Hana: In a word, three kinds (Combination (1), (2), and (5)) substantially.



Taro: It is so.

Taro: Because Combination (2) is the same as Combination 4.

Taro: And, Combination (3) doesn't exist though it will prove later.

Taro: After that, I want to add though might the superfluity because it seems to be important.

Hana: What is still?

Taro: In Virtual rule, two or more stones are in one hole, and there will be no stone in the hole when the number is even, but there will be one stone in the hole when the number is odd.

Taro: It is the same as Usual rule to remove one stone from the hole stepped over.

Hana: You should not say a strange thing suddenly.

Taro: I think that Leibnitz gave the hint of Virtual rule.

Taro: And, the essence of Peg Solitaire can be understood from Virtual rule very well.

Hana: What is Essence?

Taro: The number of stones in a hole is shown by the surplus of two in Virtual rule.

Hana: I have already heard the story when you explained Virtual rule.

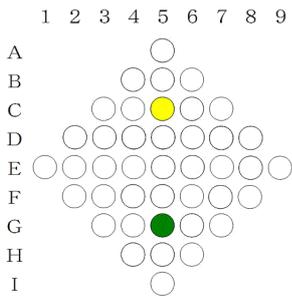
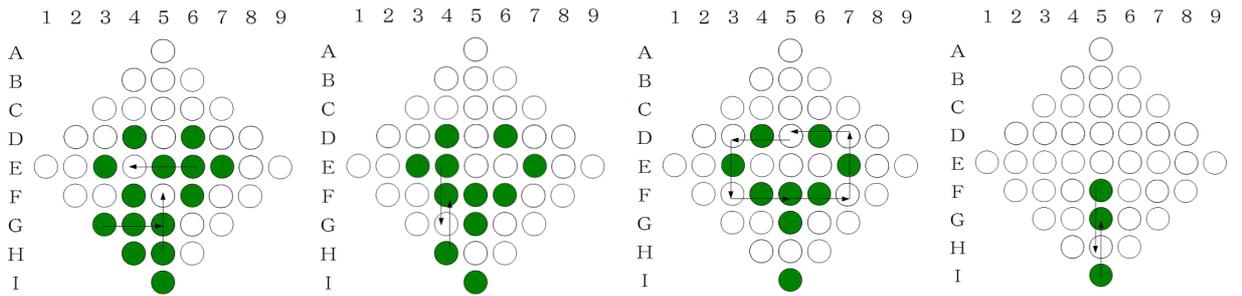
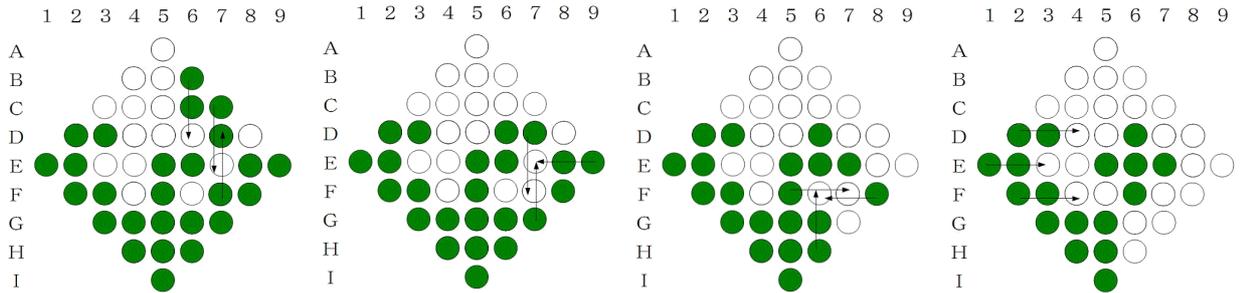
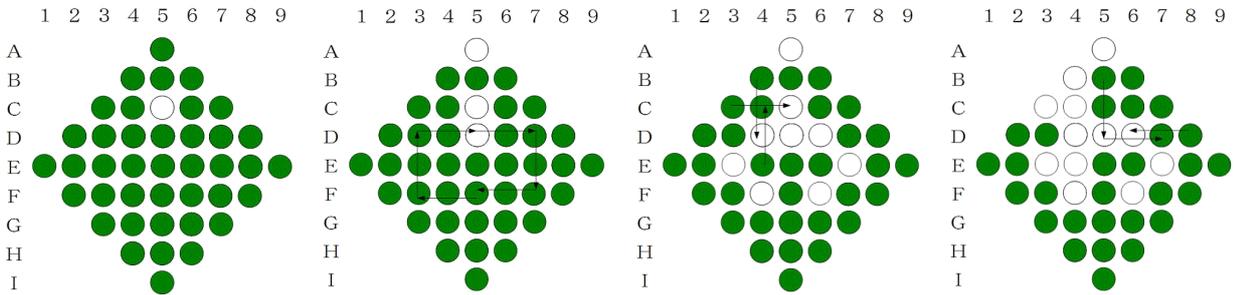
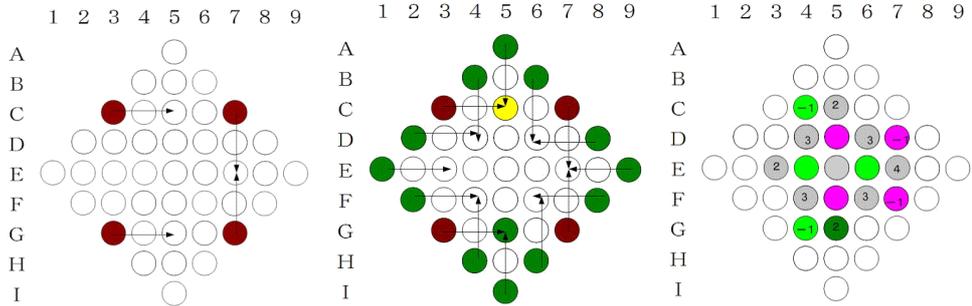
Taro: Though I had something to speak, I cannot recall it.

Example of First stone C5 and Last stone G5

Pattern 6

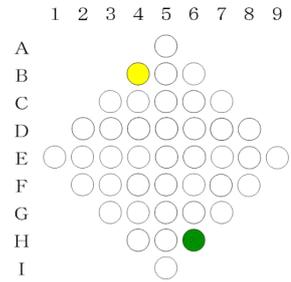
C3 → ↓ C7

C3 → ↑ G7



Impossibility of First stone B4 and Last stone H6

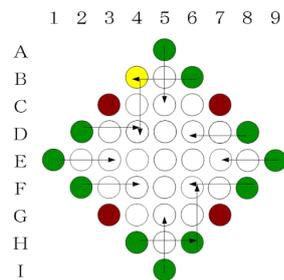
Taro: Here I would like to prove that the mobility pattern of First stone B4 and Last stone H6 does not exist as shown in right figure.



Hana: It is very difficult to prove it.

Taro: The work for the proof is considered to be very troublesome.

Hana: I think that it is surely one of the proof to examine all the mobility patterns, which pattern has possibility.



Taro: This proof looks like the proof of Four-Color Theorem.

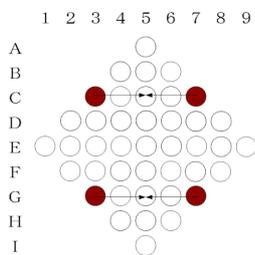
Hana: By the way, how many patterns are there?

Taro: In the following figures, all Patterns 1 to 16 of four Red stones are enumerated.

Hana: Why do not you consider the mobility pattern of Green stones?

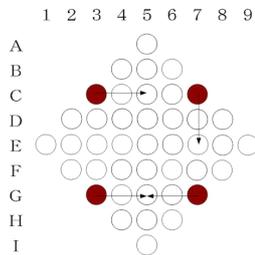
Taro: When Stone B4 is First stone and Stone H6 is Last stone, the movement pattern of Green stones is decided as shown in left upper figure.

Pattern 1



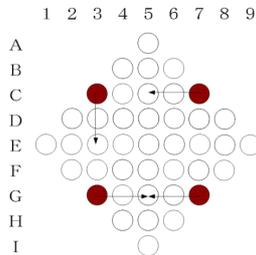
C3 →← C7
G3 →← G7

Pattern 2



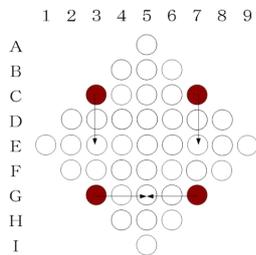
C3 →↓ C7
C3 →← G7

Pattern 3



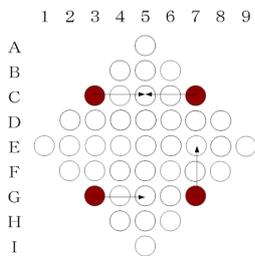
C3 ↓← C7
C3 →← G7

Pattern 4



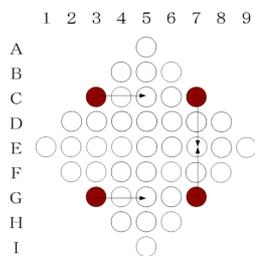
C3 ↓↓ C7
C3 →← G7

Pattern 5



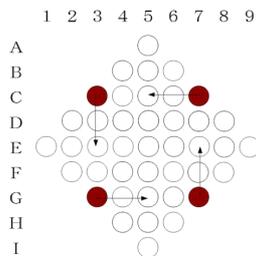
C3 →← C7
G3 →↑ G7

Pattern 6



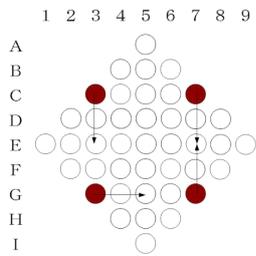
C3 →↓ C7
G3 →↑ G7

Pattern 7

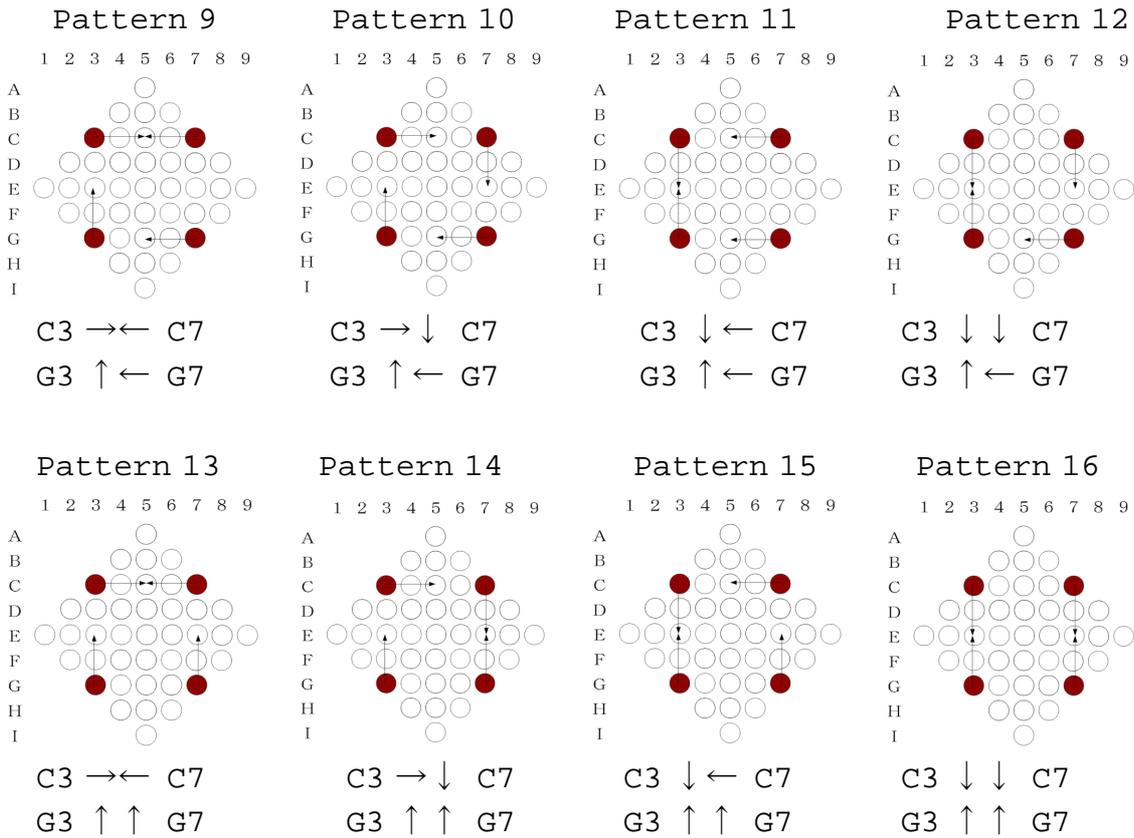


C3 ↓← C7
G3 →↑ G7

Pattern 8



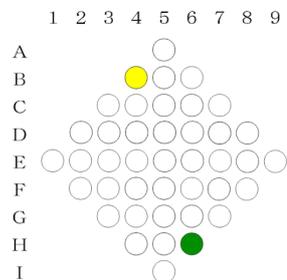
C3 ↓↓ C7
G3 →↑ G7



Taro: First stone B4 and Last stone H6 are rotary symmetry by 180 degrees.

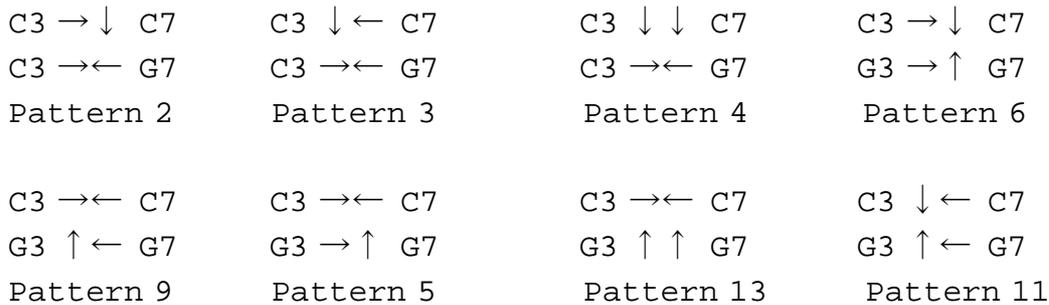
Taro: Therefore the combination of First stone H6 and Last stone B4 is possible if the combination of First stone B4 and Last stone H6 is possible.

Hana: When the pattern of right figure is turned by 180 degrees, Last stone H6 becomes First stone B4 and First stone B4 becomes Last stone H6.



Taro: Therefore only one of two patterns which are rotary symmetry by 180 degrees need to be examined.

Taro: Patterns 2 and 9, Patterns 3 and 5, Patterns 4 and 13, Patterns 6 and 11, Patterns 8 and 15, Patterns 12 and 14 are rotary symmetry by 180 degrees.



C3 ↓ ↓ C7	C3 ↓ ↓ C7	C3 ↓ ← C7	C3 → ↓ C7
G3 → ↑ G7	G3 ↑ ← G7	G3 ↑ ↑ G7	G3 ↑ ↑ G7
Pattern 8	Pattern 12	Pattern 15	Pattern 14

Taro: Each of following arrows (\leftrightarrow \uparrow \downarrow) expresses the movement direction of **Red** stone.

Taro: The plus (+) number expresses the number of stones remaining in the hole, and the minus (-) number expresses the number of stones insufficient in the hole. In addition, No number of colored hole expresses one stone in the hole. And Last stone is **Green** stone B6 with no number and First stone is uncolored hole B4.

Taro: Patterns 1, 7, 10 and 16 do not have a rotary symmetric pattern.

Hana: Only one same thing should be examined.

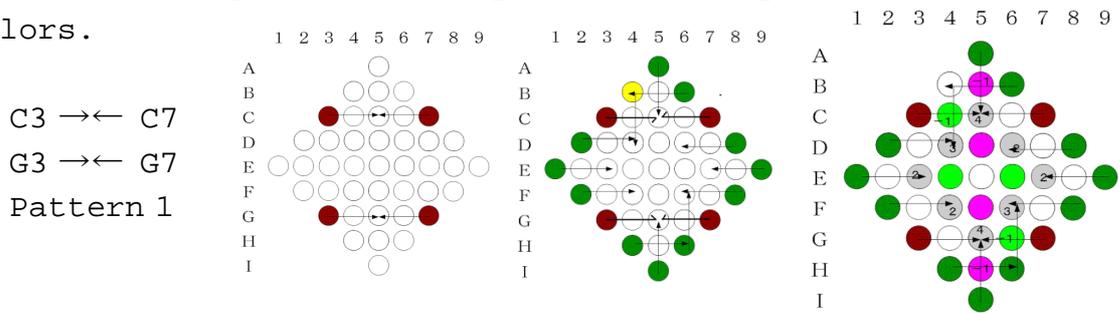
Taro: Therefore the patterns to be examined is totally ten Patterns that are four Patterns 1, 7, 10 and 16 and six Patterns 2, 3, 4, 6, 8 and 12.

Hana: In other words, the ten Patterns need to be examined.

Taro: Then let us examine from Pattern 1 sequentially.

Taro: Pattern 1 is the following figures.

Taro: The movements of the stones are illustrated with arrows in this Pattern 1, but the arrows are omitted in Pattern 2, 3, 4, 6, 7, 8, 10, 12 and 16, and only the remaining stones are separated and illustrated by colors.

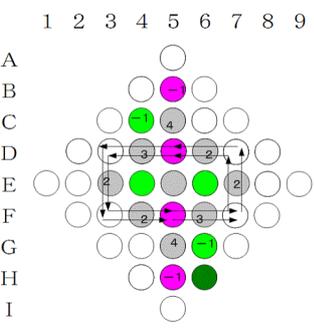


Taro: A result of the movement of stones is shown in right figure that illustrate the number of remaining stones and insufficient stones in colored holes.

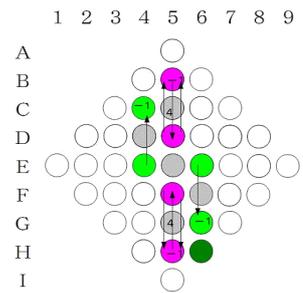
Taro: Both total numbers of Bright **Green** stones and Bright **Magenta** stones are zero together.

Hana: How are Bright **Gray** stones removed?

Taro: Then, let's really remove Bright **Gray** stones.



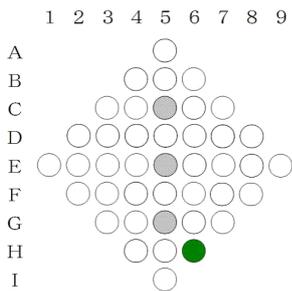
Taro: At first, either of Bright Magenta stone D5 or G5 goes around it twice as shown in left figure and comes back to original hole D5 or G5.



Taro: The result is shown in right figure.

Taro: Furthermore, Bright Green stone E4 moves to C4, and Bright Green stone E6 moves to G6. Bright Magenta stone D5 commutes to the hole B5 twice and Bright Magenta stone F5 commutes to the hole H5 twice afterwards. The result is shown in right figure.

Taro: Gray stones C5, E5 and G5 are left without being removed.



: In other words, Stone H6 cannot become Last stone with Pattern 1.

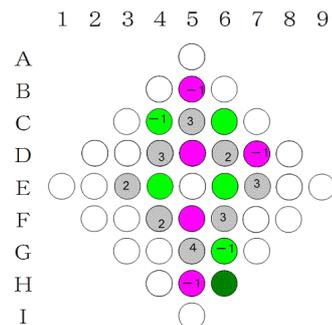
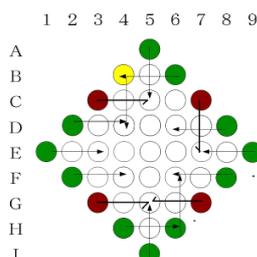
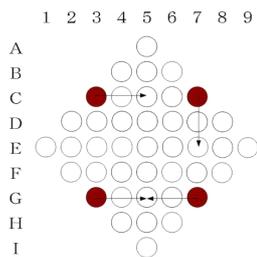
Hana: Because if Stone H6 becomes Last stone, those Gray stones are to be removed.

Taro: It is so.

Hana: Then how about Pattern 2?

Taro: Pattern 2 is the following figures.

C3 → ↓ C7
G3 → ← G7
Pattern 2



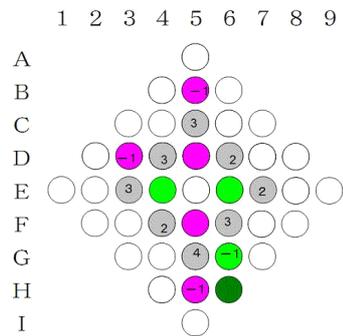
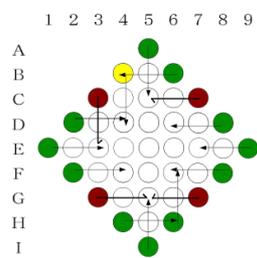
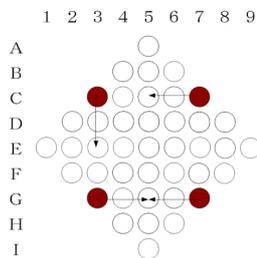
Hana: One Bright Magenta stone is insufficient, and one Bright Green stone remains.

Taro: Therefore Pattern 2 is impossible.

Hana: Then how about Pattern 3?

Taro: Pattern 3 is the following figures.

C3 → ← G7
G3 → ← G7
Pattern 3



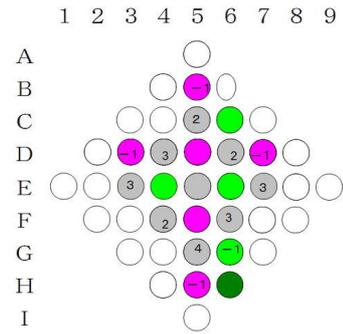
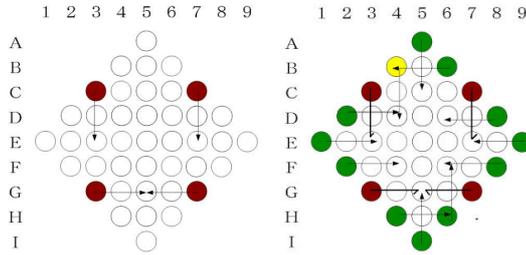
Hana: One Bright Magenta stone is insufficient, and one Bright Green stone remains.

Taro: Therefore Pattern 3 is impossible.

Hana: Then how about Pattern 4?

Taro: Pattern 4 is the following figures.

C3 ↓ ↓ C7
 G3 → ← G7
 Pattern 4



Hana: Two Bright Magenta stone is insufficient, and two Bright Green stones remain.

Taro: Therefore Pattern 4 is impossible.

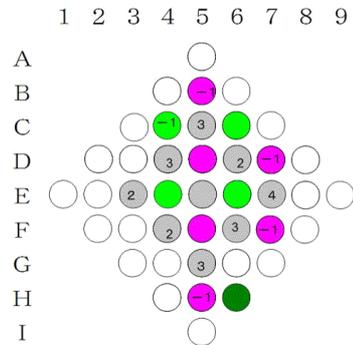
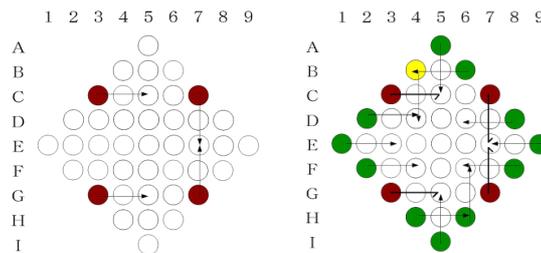
Hana: Then how about Pattern 5?

Taro: Because Pattern 5 is rotary symmetry with Pattern 3, it is impossible in the same way.

Hana: Then how about Pattern 6?

Taro: Pattern 6 is the following figures.

C3 → ↓ C7
 G3 → ↑ G7
 Pattern 6



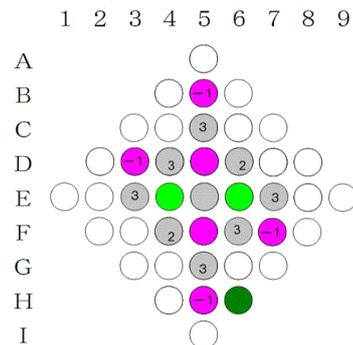
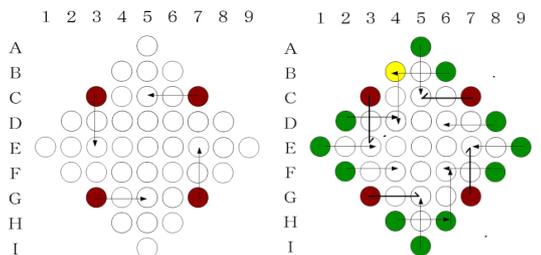
Hana: Two Bright Magenta stones are insufficient, and two Bright Green stones are remained.

Taro: Therefore Pattern 6 is impossible.

Hana: Then how about Pattern 7?

Taro: Pattern 7 is the following figures.

C3 ↓ ← C7
 G3 → ↑ G7
 Pattern 7



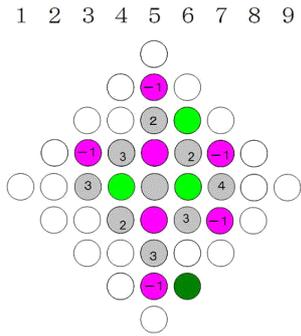
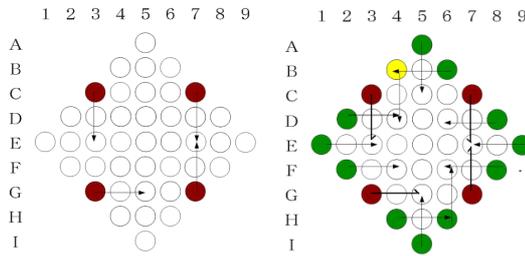
Hana: Two Bright Magenta stones are insufficient, and two Bright Green stones are remained.

Taro: Therefore Pattern 7 is impossible.

Hana: Then how about Pattern 8?

Taro: Pattern 8 is the following figures.

C3 ↓ ↓ C7
 G3 → ↑ G7
 Pattern 8



Hana: Three Bright Magenta stones are insufficient, and three Bright Green stones remain.

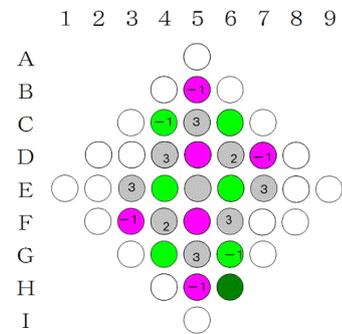
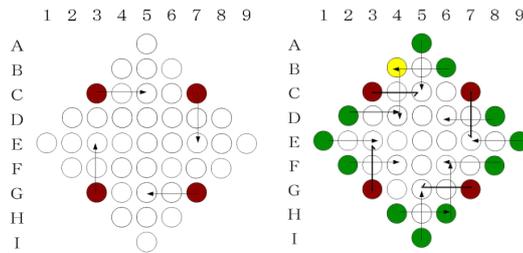
Taro: Therefore Pattern 8 is impossible.

Hana: Then how about Pattern 9?

Taro: Because Patterns 9 and 2 are rotary symmetry each other, Pattern 9 is impossible like Pattern 2.

Hana: Then how about Pattern 10?

C3 → ↓ C7
 G3 ↑ ← G7
 Pattern 10



Taro: Two Bright Magenta stones are insufficient, and two Bright Green stones remain.

Taro: Therefore Pattern 10 is impossible.

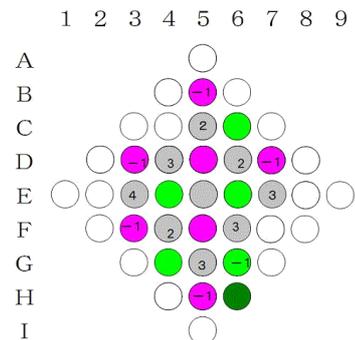
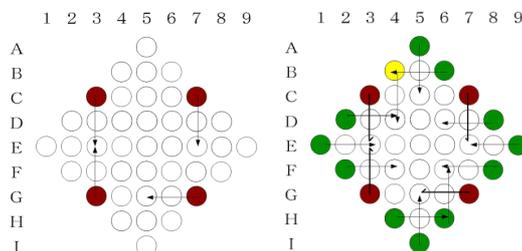
Hana: Then how about Pattern 11?

Taro: Because pattern 11 and pattern 6 are rotary symmetry each other, pattern 11 is impossible like pattern 2

Hana: Then how about Pattern 12?

Taro: Pattern 12 is the following figures.

C3 ↓ ↓ C7
 G3 ↑ ← G7
 Pattern 12



Hana: Three Bright Magenta stones are insufficient, and three Bright Green stones remain.

Taro: Therefore Pattern 12 is impossible.

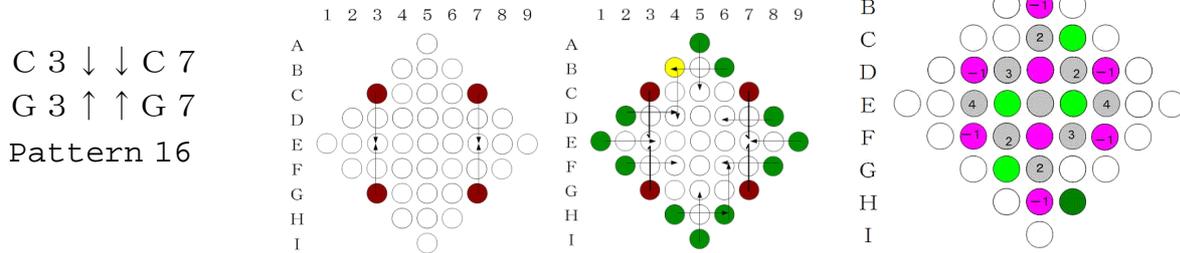
Hana: Then how about Pattern 13?

Taro: Because Pattern 13 is rotary symmetry to Pattern 4, and Pattern

14 is rotary symmetry to Pattern 12 and Pattern 15 are rotary symmetry to Pattern 8, Pattern 13, 14 and 15 are impossible in the same way.

Hana: Then how about Pattern 16?

Taro: Pattern 16 is the following figures.



Hana: Four Bright Magenta stones are insufficient, and four Bright Green stones remain.

Taro: Therefore Pattern 16 is impossible.

Hana: Pattern 1-16 were impossible entirely after all.

Taro: H6 cannot become Last stone at time of stone's First B4.

Hana: It was work to require very unrefined trouble.

Hana: Do not you know an easier proof method?

Taro: I don't know, the virtual rule is powerless in this problem.

Hana: This proof method was very complicated, and trouble entered even by virtual rule.

Taro: I think that this proof method was neither elegant nor smart.

Hana: Don't you think that examining all cases is not proof?

Taro: No, I don't think so. This proof is excellent.