Relativistic perihelion shift

Hidetomo Tohmori 2017/09/08

It is assumed that an arbitrary position in the gravitational field belongs to the inertia system which has the time axis $t'$ with the light's speed $C$ and the radial axis $R'$ with the relative speed $U(R')$ for the static system. The static system has the time axis $t$ with the light's speed $C$ and the radial axis $R$. Then, since the world space distance $dS^2$ is preserved for the coordinate transformation to the inertia system from the static system, the following formula is formed.

- $C'$: Light's speed at the arbitrary position in the inertia system (gravitational field)
- $C$: Light's speed in the static system (non-gravitational field)
- $i$: Imaginary ($i^2 = -1$)

\[
dS^2 = (iCdt)^2 = dS'^2 = (iCdt')^2 + (U(R)dt')^2
\]
\[
d\theta = 0, \ d\Phi = 0
\]

Note that the inertia system to which the arbitrary position belongs is different by the position. And also the static system is the non-gravitational field.

Since light's advancing distance is invariant without depending on the selection of the inertia system, the following equation is formed.

\[
C'dt' = Cdt
\]

Then, the following equation (6) is formed.

(6) \quad C'^2 = C^2 - U^2 , \quad U = U(R)

On the other hand, when the mass $m \ll M$ with the speed $v$ moves into the gravitational field of the mass $M$ from the infinite remote position, the following equations are formed.

\[
g = GM/R^2 : \text{Gravitational acceleration}
\]
The mass speed at the infinite remote location $v$:

The mass speed in the gravitational field $V'$:

Gravitational force acting on the mass $m$:

\[
(iCdt')^2 + (Udt')^2 + (vdt')^2 = (dS')^2 = (iCdt')^2 + (V'dt')^2
\]

\[
(Udt')^2 + (vdt')^2 = (V'dt')^2
\]

\[
V'' = v^2 + U^2
\]

\[
mV''/2 = mv^2/2 + mU^2/2
\]

\[
d(mv^2/2)/dR = 0
\]

\[
d(mV''/2)/dR = d(mU^2/2)/dR = F = mg = mGM/R^2
\]

\[
dU^2/dR = 2GM/R^2
\]

\[
U^2 = 2GM/R
\]

Then, the equation (6) is rewritten as the following.

\[
C'^2 = C^2 - 2GM/R
\]

When $C'^2 = 0$, $C^2 - 2GM/Rs = 0$, $Rs = 2GM/C^2$

$Rs$ : Schwarzschild radius

Since mass advancing distance is invariant without depending on the selection of the inertia system, the following equation is formed.

\[
V'dt' = Vdt
\]

\[
(dt/dt')^2 = 1 - U^2/C^2
\]

\[
V'' = V^2(dt/dt')^2 = V^2(1 - U^2/C^2)
\]

\[
\gamma' = 1 - e
\]

\[
2GM/V^2R = 1
\]

\[
U^2 = 2GM/R'
\]

\[
\delta = \theta - \omega
\]

$e$: eccentricity

When $\gamma' < 1$, the following expressions hold.

\[
V'(d\omega/dt) = -(dU^2/dR)\cos\delta = V'(d\omega/d\theta)(d\theta/dt)
\]

\[
d\theta/dt = V'\cos\delta/R'
\]

\[
d\omega/dt = (d\omega/d\theta)(V''\cos\delta/R') = 2GM\cos\delta/R'^2
\]

\[
d\omega/d\theta = 2GM/V'^2R = 2GM/R'V^2(1 - 2GM/R'C^2)
\]

\[
R'\cos\delta = \gamma'R
\]

\[
Rs = 2GM/C^2
\]

\[
= \cos\delta/\gamma'(1 - (\cos\delta/\gamma')(R_s/R))
\]

When $Rs/R \ll 1$, the following expressions hold.
\[
d\omega/d\theta \equiv (\cos\delta/\gamma')(1 + (\cos\delta/\gamma')(Rs/R)) \\
= \cos\delta/\gamma' + (\cos\delta/\gamma')^2(Rs/R)
\]

\[
\int d\omega \equiv \int_0^{2\pi} (\cos\delta/\gamma' + (\cos\delta/\gamma')^2(Rs/R))d\theta \\
= \int_0^{2\pi} (\cos\delta/\gamma')d\theta + \int_0^{2\pi} (\cos\delta/\gamma')^2(Rs/R)d\theta \\
(\cos\delta/\gamma')^2 \equiv 1 \\
= \int_0^{2\pi} (\cos\delta/\gamma')d\theta + \int_0^{2\pi} (Rs/R)d\theta \\
\int_0^{2\pi} (\cos\delta/\gamma')d\theta \equiv 2\pi + 2\delta\gamma e \\
\approx 2\pi + 2\delta\gamma e + 2\pi Rs/R
\]

\[2\delta\gamma e : \text{Elliptic Perihelion shift}\]

\[2\pi Rs/R : \text{Relativistic perihelion shift}\]

When \(e \equiv 0\) and \(Rs/R \ll 1\), the following expressions hold.

\[\gamma' = 1 - e \equiv 1, \quad \delta = \theta - \omega \equiv 0, \quad \cos\delta/\gamma' \equiv 1\]

\[
d\omega/d\theta \equiv 1/(1 - Rs/R) \equiv 1 + Rs/R
\]

\[
\int d\omega \equiv \int_0^{2\pi} (1 + Rs/R)d\theta = 2\pi + 2\pi Rs/R
\]

\[2\pi Rs/R : \text{Relativistic perihelion shift}\]