The ratio of prime numbers

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Abstract

The formula for generating prime numbers is derived from the formula for the sum of squares of natural numbers. The generation formula generates two series of prime numbers. Every prime number belongs to one of the two series. The ratio of prime numbers to natural numbers is calculated using the characteristics of the generation formula.

1. At the beginning

The formula for generating prime numbers is derived from the formula for the sum of squares of natural numbers. The generation formula generates two series of prime numbers. Every prime number belongs to one of the two series. The ratio of prime numbers to natural numbers is calculated using the characteristics of the generation formula.

2. Derivation of generation formula

The formula for the sum of squares of natural numbers is as follows.

$\sum_{i=1}^{k} i^2 = 1 + 2^2 + 3^2 + \cdots + (k-2)^2 + (k-1)^2 + k^2 = k(K+1)(2k+1)/6$

Since the sum of squares of natural numbers is an integer, $k(K+1)(2k+1)$ is divisible by the number 6. Since the sum of squares of natural numbers is an integer, $k(K+1)(2k+1)$ must be divisible by the number 6.

When $p_k = 2k+1$ is a prime number, $(2k+1)$ is not divisible by the number 6.

At that time, k or $k+1$ is divisible by number 3 because either of them is an even number.

As a result, the following two series exist for the generation formula of the prime number p_k.

m is a natural number

$k = 3m \qquad p_k = 2k + 1 = 6m + 1$

$k + 1 = 3m \qquad p_k = 2k - 2 + 1 = 6m - 1$

The two series are distinguished and described as follows.

$$p_k = 6m + 1 = p_{m+}$$

$$p_k = 6m - 1 = p_{m-}$$

$m =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$p_{m+} = 6m + 1 =$	7	13	19	**25**	31	37	43	49	**55**	61	67	73	79
$p_{m-} = 6m - 1 =$	5	11	17	23	29	**35**	41	47	53	59	**65**	71	**77**

$m =$	14	15	16	17	18	19	20	21	22	23	24	25	26
$p_{m+} = 6m + 1 =$	**85**	91	97	103	109	**115**	**121**	127	133	139	**145**	151	157
$p_{m-} = 6m - 1 =$	83	89	**95**	101	107	113	**119**	**125**	131	137	**143**	149	**155**

$m =$	27	28	29	30	31	32	33	34	35	36	37	25	26
$p_{m+} = 6m + 1 =$	163	**169**	**175**	181	187	193	199	**205**	211	217	223	229	**235** …
$p_{m-} = 6m - 1 =$	161	167	173	179	**185**	191	197	203	209	**215**	221	227	233 …

As described above, any prime number p_k belongs to any of the two sequences (p_{m+} and p_{m-}) generated in the order of the natural number m.

Since the natural number m is infinite, the sequence of two series also continues infinitely.

However, non-prime numbers in bold (multiples of prime numbers) are also generated.

3. Total number of non-prime numbers in the two series

The sequence of each series contains m numbers (prime or non-prime) up to the maximum number $p_{m\pm}$. Consider the total number of products of any two different numbers (prime or non-prime) in two series.

$$1 \le i \le m \qquad 1 \le j \le m$$

$$1 \quad 2 \quad 3 \ldots\ldots\ldots \; i \; \ldots\ldots\ldots \; j \; \ldots\ldots\ldots \; m$$

$$7 \quad 13 \quad 19 \ldots\ldots\ldots \; p_{i+} \; \ldots\ldots\ldots \; p_{j+} \; \ldots\ldots\ldots \; 6m+1$$

$$5 \quad 11 \quad 17 \ldots\ldots\ldots \; p_{i-} \; \ldots\ldots\ldots \; p_{j-} \; \ldots\ldots\ldots \; 6m-1$$

$$p_{i+}p_{j+} = p_{l+} \qquad l = 6ij + i + j$$

$$p_{i-}p_{j-} = p_{h+} \qquad h = 6ij - i - j$$

$$p_{i+}p_{j-} = p_{r-} \quad r = 6ij - i + j$$
$$p_{i-}p_{j+} = p_{s-} \quad s = 6ij + i - j$$

The maximum value $p_{m+}p_{m-}$ of the product of two different numbers is as follows.

$$p_{m+}p_{m-} = (6m+1)(6m-1) = 36m^2 - 1 = 6(6m^2) - 1 \cong 36m^2$$

$1 \;\; 2 \;\; 3 \ldots\ldots\ldots \; i \;\; \ldots\ldots\ldots \; j \;\; \ldots\ldots\ldots \;\; m \;\; \ldots\ldots\ldots\ldots\ldots\ldots \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots 6m^2$

$7 \;\; 13 \;\; 19 \ldots\ldots\ldots \; p_{i+} \ldots\ldots\ldots \; p_{j+} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; \ldots\ldots p_{i+}p_{j+} \,, \; p_{i-}p_{j-} \ldots\ldots 36m^2$

$5 \;\; 11 \;\; 17 \ldots\ldots\ldots p_{i-} \; \ldots\ldots\ldots \; p_{j-} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots p_{i+}p_{j-} \,, \; p_{i-}p_{j+} \ldots\ldots 36m^2$

Each series is a series of $6m^2$ numbers. Since it is a two-series, it consists of $12m^2$ numbers. If you know the number of non-prime numbers contained in these $12m^2$, you can know the number of prime numbers. The non-prime numbers contained in these $12m^2$ numbers are the product of two of the $2m$ numbers (prime or non-prime numbers) that form two series.

There are following three ways to select the two numbers.

(1) The two different numbers are selected from the m numbers in the same series.

(2) One number is selected from each series.

(3) The i of one number p_i is chosen so that $1 \leq i < m$, and the j of the other number p_j is chosen so that $m < j < 2m$.

(1) is examined.

The method of selecting two different numbers from m numbers is as follows.

$$m!/2!\,(m-2)! = m(m-1)\,/2$$

Since there are two-series, $m(m-1) \cong m^2$ non-prime numbers are generated.

(2) is examined.

There are m^2 ways to select m numbers for each of m numbers. m^2 non-prime numbers are generated.

(3) is examined.

There are 4 series as follows.

$$p_{i+}p_{j+} = p_{l+} \quad l = 6ij + i + j$$
$$p_{i-}p_{j-} = p_{h+} \quad h = 6ij - i - j$$
$$p_{i+}p_{j-} = p_{r-} \quad r = 6ij - i + j$$
$$p_{i-}p_{j+} = p_{s-} \quad s = 6ij + i - j$$

In the following, the number of non-prime numbers in one series is calculated. Since the other three series are the same, the total number of non-prime numbers is four times the number of non-prime numbers in one series.

$$0 \leq \alpha \leq 1$$

$$1 \quad 2 \quad 3 \ldots\ldots\ldots (m-n) \quad \ldots\ldots\ldots \quad m \quad \ldots\ldots\ldots (1+\alpha)m \quad \ldots\ldots\ldots \quad 2m$$

$$(m-n)(1+\alpha)m \leq m^2$$

$$(m-n) \leq m/(1+\alpha)$$

Integrating both sides with $md\alpha$ gives the following.

$$\int_0^1 (m-n)\, md\alpha \leq \int_0^1 (m/(1+\alpha))\, md\alpha$$

$$\int_0^1 (m/(1+\alpha))\, md\alpha = m^2 \ln(1+\alpha)\,|_0^1 = m^2 \ln 2$$

$$\int_0^1 (m-n)\, md\alpha \leq m^2 \ln 2$$

$$\ln 2 = 0.69314\ 71805\ 59945\ 30941\ 72321 \ldots$$

$$\int_0^1 (m-n)\, md\alpha \leq m^2 \ln 2$$

The number of non-prime numbers is $4m^2 \ln 2 \cong 2.77256m^2$.

Total number of non-prime numbers from (1) to (3) is as following.

$$2m^2 + 4m^2 \ln 2 \cong 4.77256m^2$$

Total number of prime numbers is as following.

$$12m^2 - 4.77256m^2 = 7.22744m^2$$

Therefore, the ratio of prime numbers to natural numbers is as following.

$$7.22744m^2/36m^2 = 0.20076$$

5. Conclusion

The generation of twin prime numbers is due to the fact that prime numbers are generated in two series.

The ratio of prime numbers to natural numbers is 0.20076.

This ratio is close to the result of the prime-counting function $\pi(x)$ ($\pi(x)/x \cong 0.201467$ at $x = 10^{22}$).

In the prime number function $\pi(x)$, the ratio of prime numbers is expected, but here the ratio of prime numbers is predicted.