The ratio of prime numbers

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Abstract

The formula for generating prime numbers is derived from the formula for the sum of squares of natural numbers. The generation formula generates two series of prime numbers. Every prime number belongs to one of the two series. The ratio of prime numbers to natural numbers is calculated using the characteristics of the generation formula.

1. At the beginning

The formula for generating prime numbers is derived from the formula for the sum of squares of natural numbers. The generation formula generates two series of prime numbers. Every prime number belongs to one of the two series. The ratio of prime numbers to natural numbers is calculated using the characteristics of the generation formula.

2. Derivation of generation formula

The formula for the sum of squares of natural numbers is as follows.

$\sum_{i=1}^{k} i^2 = 1 + 2^2 + 3^2 + \cdots + (k-2)^2 + (k-1)^2 + k^2 = k(K+1)(2k+1)/6$

Since the sum of squares of natural numbers is an integer, $k(K+1)(2k+1)$ is divisible by the number 6. Since the sum of squares of natural numbers is an integer, $k(K+1)(2k+1)$ must be divisible by the number 6.

When $p_k = 2k+1$ is a prime number, $(2k+1)$ is not divisible by the number 6.

At that time, k or $k+1$ is divisible by number 3 because either of them is an even number.

As a result, the following two series exist for the generation formula of the prime number p_k.

m is a natural number

$k = 3m$ $\quad\quad p_k = 2k + 1 = 6m + 1$

$k + 1 = 3m$ $\quad\quad p_k = 2k - 2 + 1 = 6m - 1$

The two series are distinguished and described as follows.

$$p_k = 6m + 1 = p_{m+}$$
$$p_k = 6m - 1 = p_{m-}$$

$m =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$p_{m+} = 6m + 1 =$	7	13	19	**25**	31	37	43	49	**55**	61	67	73	79
$p_{m-} = 6m - 1 =$	5	11	17	23	29	**35**	41	47	53	59	**65**	71	**77**

$m =$	14	15	16	17	18	19	20	21	22	23	24	25	26
$p_{m+} = 6m + 1 =$	**85**	91	97	103	109	**115**	**121**	127	133	139	**145**	151	157
$p_{m-} = 6m - 1 =$	83	89	**95**	101	107	113	**119**	**125**	131	137	**143**	149	**155**

$m =$	27	28	29	30	31	32	33	34	35	36	37	25	26
$p_{m+} = 6m + 1 =$	163	**169**	**175**	181	187	193	199	**205**	211	217	223	229	**235** …
$p_{m-} = 6m - 1 =$	161	167	173	179	**185**	191	197	203	209	**215**	221	227	233 …

As described above, any prime number p_k belongs to any of the two sequences (p_{m+} and p_{m-}) generated in the order of the natural number m.

Since the natural number m is infinite, the sequence of two series also continues infinitely.

However, non-prime numbers in bold (multiples of prime numbers) are also generated.

3. Total number of non-prime numbers in the two series

The sequence of each series contains m numbers (prime or non-prime) up to the maximum number $p_{m\pm}$. Consider the total number of products of any two different numbers (prime or non-prime) in two series.

$$1 \leq i \leq m \qquad 1 \leq j \leq m$$

1	2	3 ………	i ………	j ………	m
7	13	19 ………	p_{i+} ………	p_{j+} ………	$6m + 1$
5	11	17 ………	p_{i-} ………	p_{j-} ………	$6m - 1$

$$p_{i+}p_{j+} = p_{l+} \quad l = 6ij + i + j$$
$$p_{i-}p_{j-} = p_{h+} \quad h = 6ij - i - j$$

$$p_{i+}p_{j-} = p_{r-} \quad r = 6ij - i + j$$
$$p_{i-}p_{j+} = p_{s-} \quad s = 6ij + i - j$$

The maximum value $p_{m+}p_{m-}$ of the product of two different numbers is as follows.

$$p_{m+}p_{m-} = (6m+1)(6m-1) = 36m^2 - 1 = 6(6m^2) - 1 \cong 36m^2$$

$1 \quad 2 \quad 3 \ldots\ldots\ldots \; i \; \ldots\ldots\ldots \; j \; \ldots\ldots\ldots \quad m \quad \ldots\ldots\ldots\ldots\ldots\ldots \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots 6m^2$

$7 \;\; 13 \;\; 19 \ldots\ldots\ldots \; p_{i+} \ldots\ldots\ldots \; p_{j+} \ldots\ldots\ldots\ldots\ldots \;\ldots\ldots p_{i+}p_{j+} \;\ldots\ldots\; p_{i-}p_{j-} \ldots\ldots 36m^2$

$5 \;\; 11 \;\; 17 \ldots\ldots\ldots p_{i-} \;\ldots\ldots\ldots\; p_{j-} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots p_{i+}p_{j-} \;\ldots\ldots\; p_{i-}p_{j+} \ldots\ldots 36m^2$

Each series is a series of $6m^2$ numbers. Since it is a two-series, it consists of $12m^2$ numbers. If we know the number of non-prime numbers contained in these $12m^2$, we can know the number of prime numbers contained in these $12m^2$. Since a non-prime number is the product of two numbers contained in two series, any non-prime number contained in $12m^2$ belongs to products in the following (1)-(3). Therefore, the total number of non-prime numbers contained in $12m^2$ is the sum of number of products in (1)-(3).

(1) Product of two different numbers selected from the m numbers in the same series.

(2) Product of two numbers selected one from each series selected one from each series.

(3) Product of two numbers p_i and p_j. Then, the i of one number p_i is chosen so that $1 \le i \le m$, and the j of the other number p_j is chosen so that $nm < j < (n+1)m$.

The number of products in (1) is examined.

The method of selecting two different numbers from m numbers is as follows. There are m products of the same number, but since m is a huge number, m can be ignored compared to m^2.

$$m!/2!\,(m-2)! = m(m-1)\,/2$$

Since there are two-series and m can be ignored compared to m^2, $m(m-1) \cong m^2$ products (non-prime numbers) are generated in (1).

The number of products in (2) is examined.

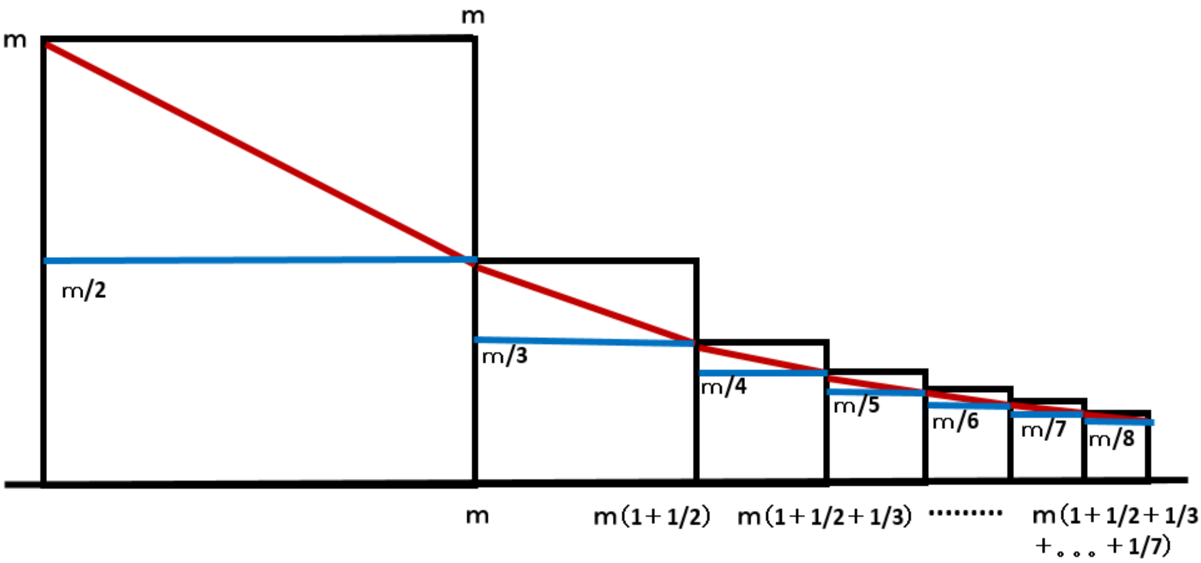
Since there are m^2 ways to select m numbers for each of m numbers, m^2 products (non-prime numbers) are generated in (2).

The number of products in (3) is examined.

There are 4 series as follows.

$$p_{i+}p_{j+} = p_{l+} \quad l = 6ij + i + j$$
$$p_{i-}p_{j-} = p_{h+} \quad h = 6ij - i - j$$
$$p_{i+}p_{j-} = p_{r-} \quad r = 6ij - i + j$$
$$p_{i-}p_{j+} = p_{s-} \quad s = 6ij + i - j$$

In the following, the number of products (non-prime numbers) in one series is calculated. Since the other three series are the same, the total number of products (non-prime numbers) is four times the number of products (non-prime numbers) in one series.



Because the product of two numbers does not exceed m^2, the following equations hold.

$$0 \le \alpha \le 1 \qquad\qquad 0 \le n \le m$$

$$1 \quad 2 \quad 3 \ldots\ldots (m-\beta) \ldots\ldots\ m \ldots\ldots\ldots nm \ldots (n+\alpha)m \ldots (n+1)m \ldots\ldots\ldots\ldots 6m^2$$

$$(m-\beta)(n+\alpha)m = m^2$$

$$(m-\beta) = m/(n+\alpha)$$

Integral with $(m/n)d\alpha$ on the right side of the above equation gives the number of products (non-prime numbers) of $p_i\ (1 \le i \le m)$ and $p_j\ (nm \le j \le (n+1)m)$ as shown in the following equations.

$$\int_0^1 (m-\beta)\,(m/n)d\alpha = \int_0^1 (m^2/n(n+\alpha))\,d\alpha$$

$$\int_0^1 (m^2/n(n+\alpha))\,d\alpha\ = m^2(1/n)\ln(n+\alpha)\,|_0^1 = m^2(1/n)\ln((n+1)/n)$$

$$\int_0^1 (m-\beta)\,(m/n)d\alpha = m^2(1/n)\ln(1+1/n)$$

The number of products (non-prime numbers) in one series is the sum of $n = 1\sim m$ as following.

$$\sum_{n=1}^{m} m^2(1/n)\ln(1+1/n)$$

The same applies to the other three series, so the number of products (non-prime numbers) in (3) is as follows.

$$1/n(n+1) < (1/n)\ln(1+1/n) < 1/n^2$$

$$\sum_{n=1}^{\infty} 1/n(n+1) < \sum_{n=1}^{m}(1/n)\ln(1+1/n) < \sum_{n=1}^{\infty} 1/n^2$$

$$\sum_{n=1}^{\infty} 1/n(n+1) = 1 \qquad \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$$

$$4m^2 < 4m^2\sum_{n=1}^{m}(1/n)\ln(1+1/n) < 2m^2\pi^2/3$$

The total number Tn of non-prime numbers in (1) to (3) is as follows.

$$Tn = 2m^2 + 4m^2\sum_{n=1}^{m}(1/n)\ln(1+1/n)$$

$$6m^2 < Tn < 2m^2(1+\pi^2/3)$$

By subtracting the total number Tn of non-prime numbers from $12m^2$, the total number Tp of prime numbers is obtained as follows.

$$Tp = 12m^2 - Tn = 10m^2 - 4m^2\sum_{n=1}^{m}(1/n)(1+1/n)$$

The ratio R of prime numbers Tp to natural numbers $36m^2$ is as follows.

$$R = Tp/36m^2$$

$$6m^2/36m^2 > R > (10m^2 - 2m^2\pi^2/3)/36m^2 \cong 0.0950$$

$$1/6 > R > 0.0950$$

Therefore, The ratio R of prime numbers Tp to natural number $36m^2$ is smaller than $1/6$ and larger than 0.0950, independent of huge number m^2.

The approximate calculation of the ratio R of prime numbers to natural numbers is explained in detail below.

First, the approximate calculation from above is as follows.

$$1 < l < m$$

$$\sum_{n=1}^{m}(1/n)\ln(1+1/n) \cong \sum_{n=1}^{l}(1/n)\ln(1+1/n) + \sum_{n=l+1}^{\infty} 1/n^2$$

$$\sum_{n=l+1}^{\infty} 1/n^2 = \sum_{n=1}^{\infty} 1/n^2 - \sum_{n=1}^{l} 1/n^2 = \pi^2/6 - \sum_{n=1}^{l} 1/n^2$$

$$\sum_{n=1}^{m}(1/n)\ln(1+1/n) \cong \sum_{n=1}^{l}(1/n)\ln(1+1/n) + \pi^2/6 - \sum_{n=1}^{l} 1/n^2$$

$$Tp/m^2 \cong 10 - 4(\sum_{n=1}^{l}(1/n)\ln(1+1/n) + \pi^2/6 - \sum_{n=1}^{l} 1/n^2)$$

$$R = (Tp/m^2)/36$$

At $l = 6$, the ratio R of prime numbers to natural numbers is $\cong 0.1376$. And at $l = 10$, it is $\cong 0.1380$. The larger l, the better the approximation accuracy.

Next, the approximate calculation from below is as follows.

$$\sum_{n=1}^{m}(1/n)\ln(1+1/n) \cong \sum_{n=1}^{l}(1/n)\ln(1+1/n) + \sum_{n=l+1}^{\infty} 1/n(n+1)$$

$$\sum_{n=l+1}^{\infty} 1/n(n+1) = \sum_{n=1}^{\infty} 1/n(n+1) - \sum_{n=1}^{l} 1/n(n+1) = 1 - \sum_{n=1}^{l} 1/n(n+1)$$

$$\sum_{n=1}^{m}(1/n)\ln(1+1/n) \cong \sum_{n=1}^{l}(1/n)\ln(1+1/n) + 1 - \sum_{n=1}^{l} 1/n(n+1)$$

$$\sum_{n=1}^{m}(1/n)\ln(1+1/n) \cong \sum_{n=1}^{l}(1/n)\ln(1+1/n) + 1/(l+1)$$

$$Tp/m^2 \cong 10 - 4(\sum_{n=1}^{l}(1/n)\ln(1+1/n) + 1/(l+1))$$

$$R = (Tp/m^2)/36$$

At $l = 6$, the ratio R of prime numbers to natural numbers is $\cong 0.1386$. And at $l = 10$, it is $\cong 0.1382$. The larger l, the better the approximation accuracy.

From the above approximate calculations from above and below, the ratio R of prime numbers to natural numbers is predicted to be in the following range.

$$0.1380 < R < 0.1382$$

5. Conclusion

The generation of twin prime numbers is due to the fact that every prime number is generated in one of the two series.

According to the number function $\pi(x)$ of prime numbers presented by the prime number theorem, the ratio of prime numbers to natural numbers is $\pi(x)/x \cong 0.176846$ at $x = 10^{25}$. As x increases further, the ratio is expected to approach 0 as much as possible.

However, as mentioned above, the ratio R of prime numbers to natural numbers is predicted to be smaller than 0.1382 and larger than 0.1380.